Theorist®

learning guide

Prescience Corporation

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Advanced users can read the Read Me file and skip ahead to "Installation Summary."

INSTALLING THEORIST

Make Backup Disks

Before you install and use Theorist, make backup copies of the disks in case of mistake or misfortune. Put your original Theorist disks away in a safe place and use the backups instead. (The originals may be necessary for upgrades.) From now on, whenever this manual mentions the Theorist disks, it means your *backup disks*.

Save Your Serial Number

Write your serial number (which is found on labels affixed to Disk 1 and the registration card) on the inside front covers of both the *Theorist Learning Guide* and *Reference Manual*. You must give this number to a Prescience technical support specialist if you need help later.

Required System Version

In order for Theorist to work properly on your Macintosh, you must have the correct version of Apple's System for your machine. Refer to the table below to determine which System version number you need.

Macintosh	System Version
Plus	6.0.2 or later
SE, II	6.0.2 or later
IIx, IIcx, SE/30	6.0.3 or later
IIci, IIfx, Portable	6.0.5 or later

As of this writing, Apple's System 7 has not yet been released, but Theorist should work under System 7 without complications.

Required Memory

Theorist requires that your Macintosh possess a minimum of one megabyte (1Mb) of RAM in order to operate. This is enough memory to perform most calculations and use all of Theorist's features without running out of memory.

Required Disk Space

Before you install Theorist and its related files on your hard disk, you must have at least one megabyte (1Mb) of space available on the disk. If not, you must eliminate some existing files to make room.

Hard Disk Installation

In order to install Theorist, you must have a basic knowledge of the Macintosh Finder. If you do not know how to create new folders, copy files, and rename files, refer to your Macintosh user's manual.

With your machine at the Finder's desktop, create a new folder on your hard disk. Rename this "Empty Folder" to "Theorist Folder".

The next step in installing Theorist is to decide which of the two Theorist applications to use.

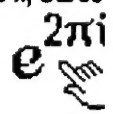
Disk 1: Application for Mac Plus, SE, Portable

Theorist®

The application on Disk 1 is for use with the Macintosh Plus, SE, Portable, or other Mac with at least one megabyte of memory. Of course, this version also runs on the SE/30 and Macintosh II systems, but it does not take full advantage of their extra speed capabilities. So if you have a Mac Plus, SE, Portable, or an upgraded older Macintosh, use the Theorist application found on Disk 1, named simply "Theorist®". If you have a Macintosh II system or SE/30, skip past the next paragraph.

Insert Disk 1 into your floppy disk drive so its icon appears on the Finder desktop. Double-click on the Disk 1 icon. A window opens, showing you the contents of the disk. Drag the "Theorist®" file icon from Disk 1 into the Theorist Folder on your hard disk. When you are done, eject the disk by dragging its icon into the trash can.

Disk 2: Application for Mac II, SE/30



Theorist®/Mac II

Disk 2 has a special version of the Theorist application, optimized for speed, for use with the Macintosh II, IIx, IIcx, IIci, IIfx, and SE/30. This built-for-speed version of Theorist takes advantage of the special math co-processors in these machines and performs mathematical operations more quickly. It also takes advantage of the 68020/-030 CPU architecture and the 68881/-882 math co-processor. This version does not run on a Macintosh Plus or SE because they are not so equipped. So if you have a Mac II or SE/30, use the Theorist application found on Disk 2, named "Theorist®/Mac II".

Insert Disk 2 into your floppy disk drive so its icon appears on the Finder desktop. Double-click on the Disk 2 icon. A window opens, showing you the contents of the disk. Drag the "Theorist®/Mac II" file icon from Disk 2 into the Theorist Folder on your hard disk. When you are done, eject the disk by dragging its icon into the trash can.

Only one application per hard disk, please.

Do not copy both applications to your hard disk, just the one you need. Having two versions of Theorist on your hard disk is not harmful, but can cause minor, annoying problems in the future. (Besides, it wastes your precious disk space.)

Accelerators

If your Macintosh is equipped with an add-in accelerator, you may be able to use the Theorist application from Disk 2 in order to take advantage of the extra speed. Your accelerator must include *both* a 68020 (or later compatible CPU) *and* a 68881 (or later compatible math co-processor). To find out whether the Mac II version of Theorist works on your machine, just try it. If your machine is incapable of running the Theorist/Mac II application, Theorist will tell you. (We recommend that you save any open files before trying this, as it may cause a system bomb.)

The next step in installing Theorist is to copy a few folders that contain examples and files that will help you do your work.

Mathematics Folder

Insert Disk 1 into your floppy disk drive so its icon appears on the Finder desktop. Double click on the Disk 1 icon. A window opens, showing you the contents of the disk. Drag the "Mathematics" folder icon from Disk 1 into the Theorist Folder on your hard disk. When you are done, eject the disk by dragging its icon into the trash can.

Graphics Folder

Insert Disk 2 into your floppy disk drive so its icon appears on the Finder desktop. Double click on the Disk 2 icon. A window opens, showing you the contents of the disk. Drag the "Graphics" folder icon from Disk 2 into the Theorist Folder on your hard disk. When you are done, eject the disk by dragging its icon into the trash can.

Animation, Color Graphics Folders

Insert Disk 3 into your floppy disk drive so its icon appears on the Finder desktop. Double click on the Disk 3 icon. A window opens, showing you the contents of the disk. Drag the "Animation" folder icon from Disk 3 into the Theorist Folder on your hard disk.

If you are using a color Macintosh, drag the "Color Graphics" folder into the Theorist Folder.

Expressionist

If you plan to use Theorist in conjunction with Expressionist, the equation editor from Prescience, there is one more step to the installation. If not, eject the disk and skip past the following subsection to "Other Files and Folders on Disk 3." (But do consider purchasing Expressionist!)

Insert Disk 3 and open its "for Expressionist®" folder. Drag the "Expr-->Theo Prefs" file into the System Folder on your hard disk. When you are done copying, eject the disk by dragging its icon into the trash can.

Click once on the "Expr-->Theo Prefs" file icon to select it, then choose Duplicate from the File menu, which creates a "Copy of Expr-->Theo Prefs" file.

If your current "Expressionist Prefs" preferences file is valuable to you, rename it to something else and/or move it out of the System Folder; otherwise place it in the trash can and choose **Empty Trash** from the **Special** menu.

Rename the "Copy of Expr-->Theo Prefs" file to "Expressionist Prefs".

Other Files and Folders on Disk 3

Other files and/or folders which may appear on Disk 3 are not directly related to Theorist and are not necessary for its operation. As such, their purpose is not described in either of the Theorist manuals; the

Read Me file (described later) should point you in the right direction to find information about these non-Theorist files. They should be ignored for now.

Installation Summary

(Power users can glance over this to determine what they want to install.)

If you followed the preceding installation instructions correctly, you did the following major steps:

- Determined which application to use and copied it
- Copied the Mathematics folder from Disk 1
- Copied the Graphics folder from Disk 2
- Copied the Animation folder from Disk 3
- At your discretion, copied the Color Graphics folder from Disk 3
- At your discretion, copied "Expr-->Theo Prefs" from Disk 3

MultiFinder

Theorist works fine under MultiFinder. The SE version's default memory allocation is 1,000K (one megabyte); the Mac II version's is 2,000K (two megabytes). You can raise or lower these numbers (with the Finder's Get Info command), depending on available memory.

Starting Theorist

Open the Theorist Folder by double-clicking its icon. When you see the Theorist application icon, double-click it to start the program.

The first time you start Theorist, it asks you to enter the last six digits of your serial number. It also asks you for your name and the name of the organization you are working for (if any). You will not be able to use Theorist without entering this information. After you enter this information the first time, you do not need to enter it again.

Latest News

Read Me

The "Read Me" file has late-breaking news and information about Theorist. With your machine at the Finder's desktop, insert Disk 1 and open its disk icon. Double-click the "Read Me" file icon, which launches Theorist (if it is not already running) and opens the "Read Me" file. With the mouse pointer, pull down the **File** menu and choose **Print...**, then choose the appropriate print options to print a copy on your printer. Carefully read the printed document, as it may have information that affects the *Theorist Learning Guide* and/or the *Theorist Reference Manual*.

Getting Started

INTRODUCTION

If you have a question or need further information about a feature, check the Theorist Reference Manual.

Each of the following tutorials assumes you know how to perform some basic tasks with Theorist. The first tutorials assume little, but as you progress, more and more knowledge is assumed. Most of the tutorials, especially the first few, are excruciatingly detailed for your benefit. It may seem like you have to do too much work for a simple operation. The more advanced tutorials are less detailed because they assume you learned a lot from the earlier tutorials. You can also learn shortcuts to make your work quicker.

Before you begin the first tutorial, here are some things you must know...

When you start the Theorist application, you get a blank, untitled document which is a Theorist notebook. Each tutorial assumes you are starting it with a new notebook, with its default settings. To create a new notebook, choose **New Notebook** from Theorist's **File** menu.

One of the ways you enter expressions into Theorist is with keyboard formulas. If you have ever used a programming language or computer spreadsheet, you already know how to do it. For less experienced users, whenever the manual says something like "Type: `2*x+y-z`", type in those characters exactly as they appear, being careful to differentiate between upper and lower case characters When a formula appears in this typeface: "`x*y`", you are the one who must decide when to use Shift for certain characters.

If the manual tells you to type the following,

`2*x=y` Return `(z^3)=2*x`

...in this case, do not type out the word R-e-t-u-r-n, but instead press the Return key.

Sometimes the manual may instruct you to type "`x*` Option-G", in which case you would type the *x* and the multiplication symbol, * (Shift-8), then hold down the Option key as you press G. (Do not type the dash.) When the manual instructs you to press a keystroke in this typeface: "Command-9", do not press Shift or any keys other than

those explicitly stated. For example, if the manual tells you to press Option-B but you press Option-Shift-B, Theorist will create the "⟨" symbol rather than the Greek letter β which was intended.

The following table shows how a keyboard formula may appear in the manual, how to type it, and the result is has in Theorist.

Formula	How to Type It	Result
`g`	Press the G key	g
`G`	Hold down Shift as you press G	G
`George`	Press Shift-G, then E, O, R, G, and E	George
Option-G	Hold Option, press G (*no* Shift!)	γ
Option-Shift-G	Hold Option and Shift, press G	Γ
Command-G	Hold Command as you press G (*no* Shift!)	(Graph)
`x^y` Esc `z`	Press X, Shift-6, Y, Escape, and Z	$x^y z$
2 space `x`	Press 2, then space, and then X	$2x$
Option-minus	Hold Option as you press the - key	(Negation)
Command-slash	Hold Command as you press the / key	(Division)

The Command key is the one with the clover (⌘) and/or apple (Ꞓ) on it. The Esc (escape) key is located in the upper-left region of most keyboards.

Do not press the Return or Enter keys unless you are specifically told to do so. If you have used command-driven or line-oriented software programs before and you are accustomed to pressing Return or Enter at the end of an input line in order to execute the command, you must unlearn that tendency in order to use Theorist properly. When you press Return in Theorist, it creates a new place for an expression. Pressing Enter has another entirely different effect. Remember: do not press Return or Enter at the end of a line.

In many cases the example keystrokes the manual provides are longer than necessary and they have many parentheses. This is to make the keyboard formulas consistent and easier to understand.

While entering formulas, the standard Macintosh text-editing conventions are used. However, because of the nature of WYSIWYG equations, there are differences, which you will become aware of as you read further in the manual. When you type an alphanumeric character (that is, a letter or number) it appears in the equation at the insertion point indicated by the blinking cursor. The arrow keys move the cursor and the Tab key also changes the selection, when possible. Escape takes you out of the current selection to the next hierarchical level in the expression. You make selections by dragging the mouse pointer, and you can make multiple selections by holding the Shift key as you

drag. The Delete (or Backspace) key generally erases the selection or the character to the left of the cursor. It also changes the selection, so be careful when typing Delete repeatedly, or you may lose large pieces of expressions. If you make a mistake entering a formula, try the **Undo** command from the **Edit** menu. Until you become more experienced at editing, it might be easier to correct mistakes by deleting the whole equation and starting over. By the end of this manual, you will be a pro at entering and editing equations with Theorist.

QUICK SOLVE AND GRAPH

Start up Theorist by double-clicking on its application icon in the Finder.

Type "`z=1.5^x` Esc `*sin(y)`", which gives you:

$$\square\ z = 1.5^x \sin(y)$$

Select the *x* by dragging over it. Choose **Isolate** from the **Manipulate** menu.

$$\triangle\ x = \log_{1.5}\left(\frac{z}{\sin[y]}\right)$$

Select the *y* in the result. Choose **Isolate** from the **Manipulate** menu. When the dialog comes up, click **PreDefined** or press Return to accept it as the default.

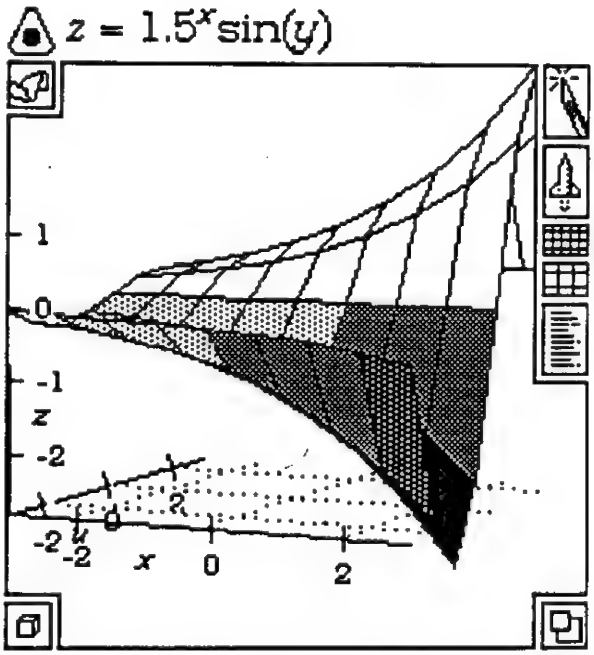
$$\triangle\ y = \arcsin\left(\frac{z}{1.5^x}\right)$$

Select the *z* in the result and, once again, choose **Isolate** from the **Manipulate** menu.

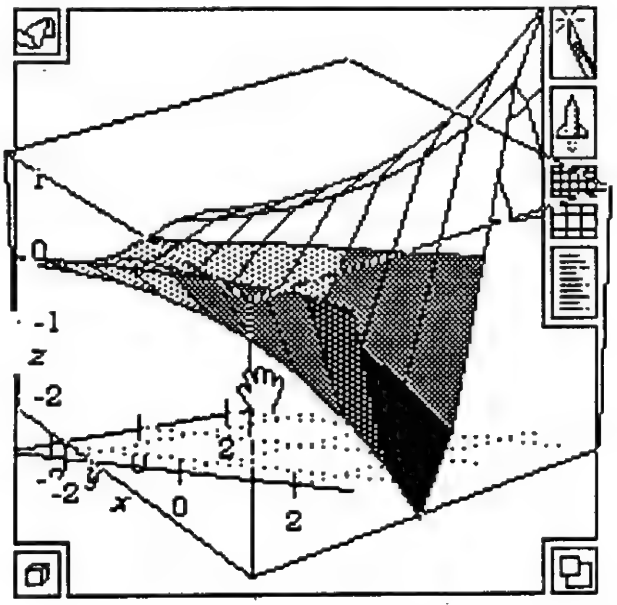
$$\triangle\ z = 1.5^x \sin(y)$$

Notice that this is what we started with in the first place.

With the cursor anywhere in the latest equation, pull down the **Graph** menu, and from its **z = f(x,y)** submenu, choose **Illum. 3D**, or press Command-U. When the dialog comes up, click **OK** or press Return.



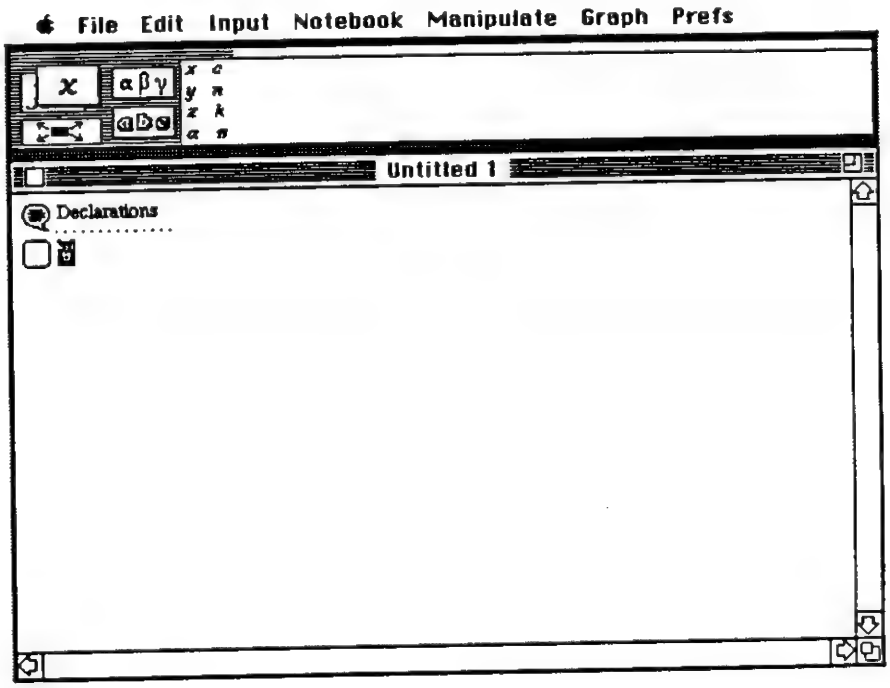
Next, click the mouse pointer on the graph surface and hold the button down as you move the pointer. Notice how the cube moves as you drag the hand to rotate the graph.



Try several different directions, releasing the mouse button to redraw the graph in the new view.

BASIC CONCEPTS

When you start up the Theorist application, the screen looks like this:



The window—named "Untitled 1" in this example—is your notebook, or work area. It is analogous to a word processor document or a spreadsheet program file but is perhaps most like an outliner document.

On the above screen, this window shows two Theorist "propositions." The first one, "Declarations," is a comment, a heading for the notebook. You can type anything into it and it is ignored by Theorist in calculations. Go ahead and do this now: click the pointer anywhere in the word "Declarations" and when the blinking cursor appears, start typing. (If you are wondering why it is named Declarations and what the dots below it are for, this particular comment has some special declarations hidden inside of it, but we will discuss this later in the "Name Declarations" tutorial.)

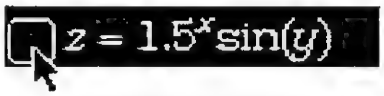
The second proposition is an assumption—an equation that you type to tell Theorist something. Notice that the comment has a quotation bubble icon to its left, whereas the assumption has a rounded-corner square icon. These proposition icons represent two types of propositions. There are many other types, discussed later in this

manual.

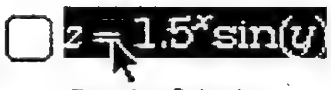
You can make more assumptions by pressing Return and you can make more comments by pressing Shift-Return. Press Shift-Return now and a bubble icon appears with a blinking cursor. Type "`hello`". Press Shift-Return. Because the blinking cursor is in a comment, Shift-Return adds a line to the current comment rather than creating a new comment as it would do with a different selection. Type "`world`".

Use the mouse pointer to select the word "world." You can either drag over it, double-click it, or Shift-click it to make the selection. Press Delete (or Backspace) and notice the selection disappear.

You can select propositions by clicking on their icons and you can delete any proposition by selecting it and pressing Delete on the keyboard. A proposition selection is different from an equation selection and manipulations have different effects on these different selections. Press Return and type "`z=1.5^x` Esc `*sin(y)`". Click on the equals sign to make an equation selection. Click on the square assumption icon to make a proposition selection. Notice the difference in the way the two selections are highlighted.



Proposition Selection



Equation Selection

Entering Equations

Theorist gives you two methods of entering expressions and equations: keyboard entry and palette entry. Keyboard entry consists of typing formulas as in a programming language or spreadsheet; it may be hard to learn for some, but it is fast. Palette entry lets you use the mouse to click on palette buttons on the screen that represent various symbols and operators. Both methods are reasonably fast, but the keyboard method is faster for experienced users.

Palette Entry

Theorist has a rectangular palette at the top of the screen for entering equations. Like a window, you can drag it to other locations on the screen and you can have it showing or hidden. Actually, it is two palettes in one. The variables palette, labeled *x* on the left, is for entering names for variables and constants. The function palette, labeled *f(x)* on the left, is for entering operators and functions. Clicking anywhere on the big *x* or *f(x)* buttons toggles between the variable and function palettes. Do this now and notice the palette switching between function and variables modes.

When you click on a palette button, the desired symbol, function, or operator appears on your notebook and the palette switches modes, anticipating your next entry. For example, click on the *a* button from

the variables palette; it switches over to the function palette. Click on the multiplication button $x * y$; it toggles back to the variables palette. Click the y button. The process continues. What you just clicked is the palette equivalent of the keyboard formula `a*y`.

The palette method of entering expressions is affected by a special option, described below.

Keyboard Entry

Theorist has two styles of entering expressions from the keyboard. The default style works in much the same way expressions are entered into Expressionist, if you are familiar with that program. The other style, 'Fortranish,' works like—you guessed it—the FORTRAN programming language. The **Fortranish** item on the **Prefs** menu lets you choose which of the styles you wish to use.

The two styles are very similar. You can type in expressions in a manner similar to the way you would type them into a computer spreadsheet or into a Pascal program. Addition and subtraction are represented by the familiar "+" and "–" keys; multiplication is "*" and division is "/". For a complete listing, see the Theorist Reference Manual.

For example, try typing the following expression:

```
(a^2)=(x^2)+2*x*y+y^2
```

It appears on your notebook as:

$$\square\ (a^2) = (x^2) + 2xy + y^2$$

Choose **Clarify** from the **Notebook** menu to eliminate the unnecessary parentheses, which gives you:

$$\square\ a^2 = x^2 + 2xy + y^2$$

Notice that Theorist figures out what to do as you type. For instance, when you type "`(x^2)+2`" it gives you $x^2 + 2$. To change this grouping, type as you would in a spreadsheet or a FORTRAN or BASIC program without parentheses. If you want x^{2+2}, type in "`x^2+2`". Try this by pressing Return and typing in this equation:

```
y=x-1/x+1
```

It looks like:

$$y = x - \frac{1}{x+1}$$

But if you type:

```
y=(x-1)/x+1
```

...then you get:

$$y = \frac{(x-1)}{x+1}$$

Notice that as you type the expression, the "(" makes both the opening and closing parentheses ( ). When you type the ")" closing parenthesis, you tell Theorist you are done typing the contents of the parentheses and it places the blinking cursor to the outside of the parentheses. With this scheme, you never have mismatched parentheses.

Sometimes, Theorist changes parentheses to brackets and braces, and vice versa. Do not be alarmed; this has no effect on how the equation is manipulated; it only affects the screen display. You may even type []'s, {}'s, and ()'s at will, as they are interchangeable.

In many cases, you can enter expressions without the unneeded parentheses. The Escape key moves the selection (the blinking cursor) so parentheses are not necessary. For example, if you wanted to create the expression $x^2 + 2$ with the Fortranish option off, you could type:

```
(x^2)+2
```

Or you could type:

```
x^2 Esc +2
```

Notice how as you type the Escape key, the selection moves from the superscript area back down inline with the x.

Internally, Theorist knows what is enclosed in what and never needs to see the parentheses to understand it. They are there for your benefit in your efforts to enter expressions.

Theorist removes parentheses that are unneeded, especially in exponents, subscripts, and the tops and bottoms of fractions. Even though Theorist doesn't always show parentheses in these, you generally have to use parentheses while typing them in or Theorist misinterprets your

meaning. When you need to enclose something in parentheses after it has been typed in, select it whole and type an open parenthesis: "(".

If and when you become a keyboard entry ace, you may want to get rid of the palette at the top of the screen. Choose **Show Palette** from the **Prefs** menu to un-check the menu item and hide the palette.

There are more details to entering complicated equations; examples in the following Tutorials section show you how. The Advanced Equation Entry chapter in the Theorist Reference Manual has everything you need to know about entering equations into Theorist.

Exercise

Try to create this equation:

$$a = \frac{1}{\frac{1}{x} + \frac{1}{y}}$$

Hint: It doesn't need to look exactly the same, just as long as it has the same meaning. Effective use of parentheses is the key to success.

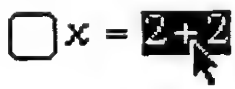
Selecting

You make a selection to later instruct Theorist to change, delete, or otherwise modify the selection by some action. Most manipulations and commands need a selection in order to work.

As you may have realized by now, you can select any part of any equation or expression simply by clicking and dragging with the mouse. However, you can't select an op, such as = or +, by itself; you must include the terms it operates on, such as $x = y$ or $a + b$.

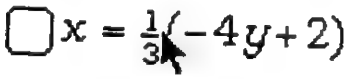
There are shortcuts to dragging. For instance, you can select an entire equation just by clicking on its equal sign:

Another shortcut is to select a sum of terms by double-clicking on an addition sign in the expression:



You may select integrals by clicking on an integral sign:

Double click here:



...to select this term:

Select cross-products by clicking on the cross product symbol:

By now you should have an idea of how selection shortcuts work. The general rule of thumb is to click or double click on the symbol or operator. Try experimenting with selection shortcuts by creating large equations and clicking various locations within the equation.

Note: You can't make multiple selections in comments.

You can make multiple selections by Shift-clicking, which involves holding down the Shift key as you click down. Try making two or three selections in different equations in your window. Then, try typing something in. Whatever command you choose or whatever you type affects all the individual selections simultaneously.

Every time you click without Shift, you start another selection, de-selecting your previous selection. After the first, every Shift-click adds to the set of selections. If you try to add another selection that is inside (or another part of) an existing selection, you de-select both (just like selecting icons in the Finder).

Modifying Equations

Modifying equations can seem to be a bit counter-intuitive.

If you are trying to delete something, it is best to select it, and press Delete one or more times to get rid of it. Sometimes it takes a few deletions; frequently Delete leaves a question mark as a place-holder, which gets deleted with the next keystroke. As you continue to type Delete, it gradually unravels the whole equation or expression.

Type "`y+z+3x^2`". Select the 2. Press Delete. Keep on pressing Delete

until you have nothing left. Repeat as often as necessary in order to understand it.

If you are trying to replace something, it is best to select what will be replaced, and then type in the new expression.

Create a new assumption and enter the same expression again. Select the 2 and type "`5+n`". Select the x and type "`sqrt(2)`". Select the $y + z$ and type "`sin(x)`".

$$\square \sin(x) + 3\sqrt{(2)^{5+n}}$$

When augmenting an expression with new parts, the new parts almost always start with an operator. In this case, select what you want to enclose in the op and then type it or click it from the palette.

Create a new assumption and type "`3x^2`". Select the 3. Type "`+a`". Then, press Delete twice to get rid of it.

Select x. Type "`+a`", then press Delete twice to get rid of it.

Select the 2. Type "`+a`", then press Delete twice to get rid of it.

When you replace a negated expression, you must press Delete before typing the new expression. For example, select the 2 and type "`-a`". press Delete *three* times to get rid of it. Two Deletes are not enough to get rid of the negation. Select the 2, press Delete, and type "`-a`".

USING NOTEBOOKS AND FILES

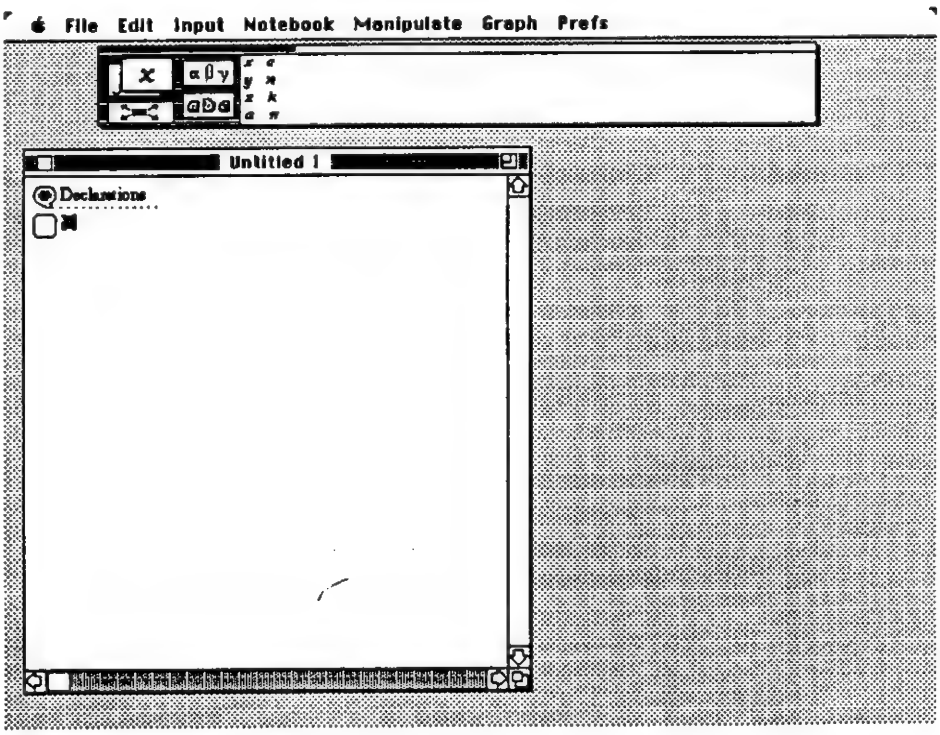
Here you learn about notebooks, and how to use and save them for later use.

Notebook Window Close Box

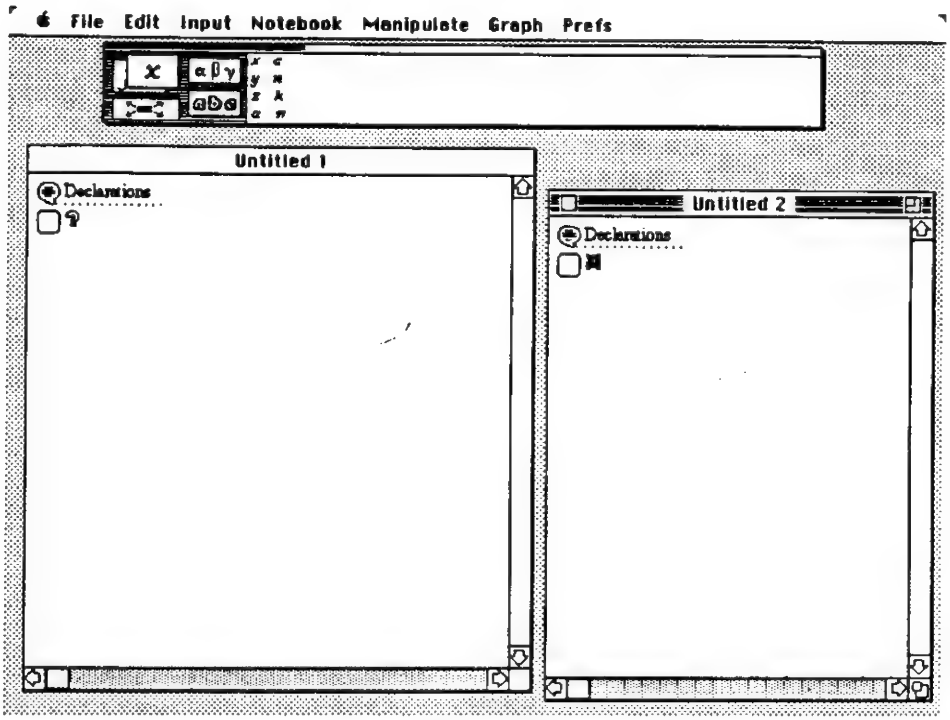
Notebook Window Resize Box

If you have any open notebook windows, close them. (To close a notebook window, click on the close box in the upper-left corner of the window.)

Create a new notebook by pulling down the File menu and choosing the **New Notebook** item, or by pressing Command-N. Use the window's resize box (in the lower-right corner) so that it occupies only half of the screen.



Create another new notebook and arrange it so that takes up the other half of the screen. (Resize it smaller, then grab the window where it says "Untitled 2" and drag to position it in the desired location.)



Each notebook is self-contained and independent of other notebooks.

Click the first notebook's window bar to make it active, then, with the question mark selected, type "**x**=5", press Return, and type "**x**". Select *x* in the second proposition and choose **Calculate** from the **Manipulate** menu. The result is 5. Now click the other notebook window to make it active. Make sure the question mark is selected and type "**x**". Select and Calculate it—you do not get the answer 5 because definitions in one notebook do not apply to other notebooks. Put simply, a given notebook does not know about anything else declared or defined in other notebooks.

The two notebooks you have are both named 'Untitled' along with a number. Whenever you create a new notebook, Theorist names it 'Untitled #' until you change the name by saving the notebook as a file on your disk.

Click on the close box of the first notebook. Theorist asks you if you want to save it. Respond in the affirmative by clicking the **Yes** button or pressing Return. Theorist then puts up the standard "Save As" dialog, which lets you specify a name for your notebook file and specify in which folder on which disk you want the notebook to be saved. Type in

Be sure to type a space between the name Notebook and the number.

the name "Notebook 1". Make sure that the folder it's saved in is the same as the folder your Theorist application is in. Click the **Save** button or press Return.

For the second notebook, pull down the **File** menu and choose the **Save As Stationery...** item and save it as "Notebook 2". Close it.

Now, no notebooks are open. Let's change that: pull down the **File** menu and choose the **Open Notebook...** item, or press Command-O. Choose the Notebook 1 file, and the notebook opens on the screen. Before you do anything else, click on its close box. Because you made no changes, it closes without asking if you want to save it.

Open Notebook 1 again. Type:

Return `z=x+y`

Select and Calculate your new equation. Now that you have made changes, close the notebook. Theorist asks whether you want to save it. You do not have to specify a name and location this time, since Theorist remembers its name and where to save it. The old Notebook 1 is replaced by the new Notebook 1.

Open Notebook 2 this time. You may recall that you saved it as "stationery," which is something special. Make some changes and click on the close box. Theorist asks if you want to save it as a notebook (yes, you do). Instead of replacing the Notebook 2 stationery file, Theorist asks you for a new name. Type in "Notebook 3" as the name and press Return. Exit the Theorist application to return to the Finder by choosing **Quit** from the **File** menu or by pressing Command-Q.

At the Finder (desktop level), make sure **by Icon** is checked on under the **View** menu. Open the Theorist Folder where your notebooks are saved and look at the icons:

Note the first and third look like notebooks while Notebook 2 looks like a pad of paper, because it is a Theorist stationery file. Stationery is like a pad of your own personal letterhead which already has your name and address on it. When you want to write a letter you simply tear off a piece of paper. In the case of Theorist stationery, you make a stationery notebook with all your favorite definitions and rules in it. Then, when you want a new notebook, you can open the stationery file and the resulting notebook has your own definitions.

Creating a new notebook (**New Notebook** under the **File** menu) is equivalent to opening a stationery file with minimal definitions in it. There is no way to change the definitions that **New Notebook** generates, so for different definitions use stationery.

It does not matter how you open the stationery file; you can open it from the **Open Notebook...** menu item or you can start up Theorist by double-clicking on a stationery icon from the Finder.

There is another way to open a notebook file. When you launch the Theorist program (do it now), it automatically opens all files named "Notebook *n*", where *n* is an integer from 0 to 10. You have the notebooks Notebook 1, Notebook 2, and Notebook 3 open, and all you did is double-click on the Theorist icon.

When you close a numbered notebook, Theorist remembers its special status and automatically saves it without asking you. In Notebook 1, make some changes and close it. Theorist saves it automatically.

These special, numbered notebooks that open automatically are for notebooks that you consider 'home base,' or ones with which you do a great deal of work. Through multiple launches and quits of Theorist, the notebook contents are always preserved.

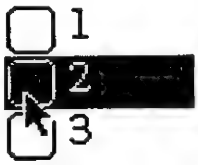
However, since Notebook 2 is a stationery file, Theorist prompts you for a new name when you save it. So if you want, you can make a stationery file with your favorite declarations, and name it "Notebook 0". Then, whenever you start Theorist, you get a notebook of your most commonly used declarations instead of an Untitled notebook with bare-bones definitions. You can stop this automatic opening by simply changing the file names to something else.

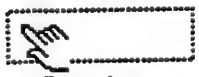
NOTEBOOK STRUCTURE

Theorist allows you some degree of flexibility for arranging the internals of notebooks.

Create a new notebook and type "1" into the empty assumption. Press Return and type "2". Repeat this process through ten.

Click on the second proposition icon so it is selected.





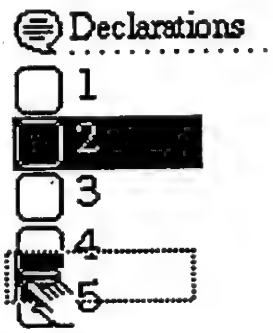
Dragging a Proposition Outline with the Pointing Hand Cursor

With the pointer over the icon, hold down the Command key. Notice how the pointer turns into a pointing hand. With Command still held, click and drag the second icon about the screen. (Once you start dragging, you may release the Command key.) Note that an outline of the proposition is dragged around.

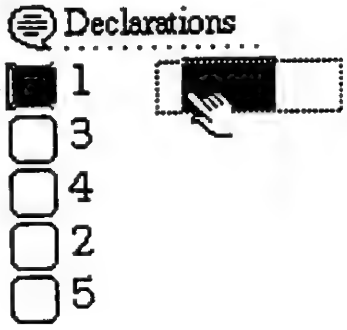
Also notice that at certain locations on the screen, highlights appear under the hand. These are locations where you can drop whatever the hand is holding. Most places between propositions can accept a new proposition.

Drop 2 between 4 and 5. Then take 10 and move it to the right of 1.

Now put them all back in order again.

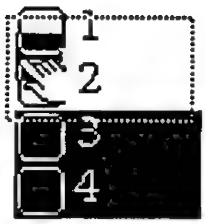


Moving Proposition #2



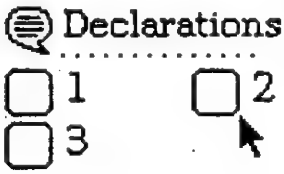
Moving Proposition #10

Now that everything is back in order, we'll move multiple propositions at a time. Select 3 and 4. To do this, click 3's icon to select it, then hold the Shift key as you click 4's icon to select it. Hold down the Command key and drag 3 and 4 (as a group) to a new location between 1 and 2.



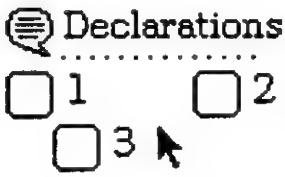
Once again, rearrange all of the propositions so they are back in order.

Take the second proposition and move it to the right of the first.

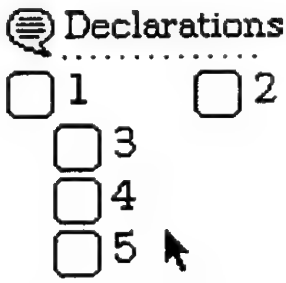


Every Theorist proposition can have another to its right, called an Inline proposition. (More properly, 2 is an 'inline' of 1.) This is handy, for instance, for putting several short equations all on one line, or for putting two graphs side by side.

Now grab proposition 3 and move it up slightly and over to the right so it is indented below 1.

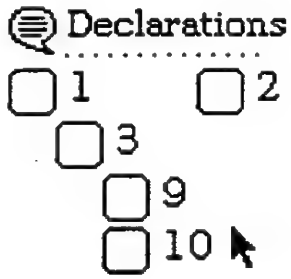


In this configuration, 3 is a 'daughter' of 1. Now, move 4 directly under 3, and 5 directly under 4.



Configured like this, 3, 4, and 5 are all daughters of one. They are also referred to as 'sisters' of one another. In fact, 1 and 6 through 10 are all considered sisters of each other.

Next, make 9 a daughter of 3 and move 10 immediately below it:



Daughters can go on for as many levels as you can fit on your screen, and beyond, with the aid of the scroll bars.

Now click on the first proposition icon and its daughters are also selected. Choose **Collapse** from the **Notebook** menu (or double-click the selected icon). All of 1's daughters collapse under 1. (Note that 2, a sister of 1, is still visible.) The dots next to the 1 tell you how many daughter propositions are buried beneath it; there is one dot for each, plus an extra three thrown in by Theorist just for good measure.

With the first proposition selected, un-collapse it by choosing **Expose** from the **Notebook** menu. (Or, you can double-click it.)

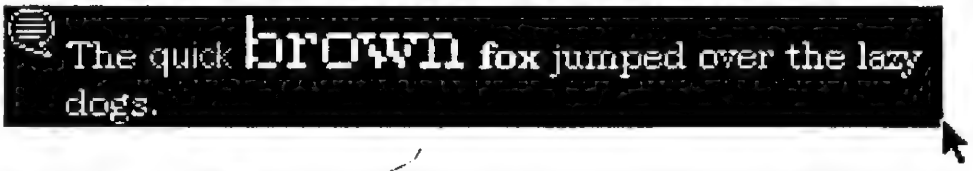
Theorist's ability to collapse certain information is a good way to hide information. For instance, expose the first proposition in the notebook, "Declarations." The name declarations for different variables appear. (Don't worry about them now. You will explore them at length in future tutorials.) Collapse the declarations.

Press Shift-Return, which makes a special proposition called a comment. You can type normal text into a comment and Theorist won't try to interpret it as an equation. Like a word processor, you can click, drag, delete, et cetera. Type "`The quick brown fox jumped over the lazy dogs.`" Select the word "quick". From the **Comment Font** submenu on the **Edit** menu, choose a favorite font of yours (other than the current font). In the sentence, select "brown". From the **Comment Size** submenu, choose a larger size, such as 24 points. Select the word "fox" and with the **Comment Style** submenu, make it boldface. Finally, if you happen to be using a color-equipped Macintosh, select the word "jumped" and change its color with the **Comment Color** submenu.

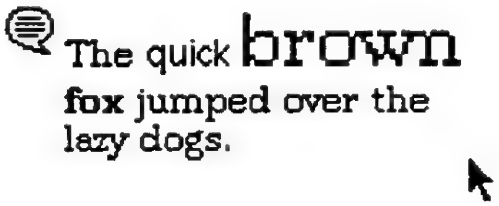
The comment text is formatted inside a rectangle, invisible until you

click on it. To find it, click on the comment proposition icon:

to select it. Find the size box in the lower right corner:



and drag it to resize the comment. All you are really adjusting is the column width.



From the **Prefs** menu, choose the **Comment Default...** item. A dialog appears, in which you may use the pop-up menus to set the default font and size for comments. Change them to something of your own choice. When you create a new comment (press Shift-Return), the text you type into it is in the new font and size:

A Brief Tour of Theorist

Simple Calculations

Here we learn how to do simple calculations with Theorist. We explore the Calculate and Simplify manipulations, as well as some of the central ideas behind Theorist derivations.

Theorist does mathematics mainly in three ways:

1. Numerically (calculations)
2. Symbolically (equations)
3. Graphically (graphs)

Theorist has in it all the functions of a pocket calculator. Therefore it can take any equation you give it and crunch all of the numbers together to get an answer that is a number. The problem with this method is that sometimes you want a more abstract answer, an equation.

Equations can be manipulated symbolically, doing algebra, calculus, et cetera as you would on paper. Theorist can be much more powerful doing things symbolically, but the power comes at the price of more computer overhead (memory and processing speed). If your task is too big or too complicated, you may end up having to do it numerically. Some problems can't be solved symbolically.

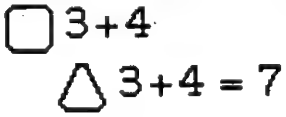
Theorist can draw graphs of mathematical relationships. It does this with high-intensity calculations. More on this subject later.

Calculate and Simplify

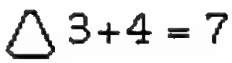
Type in "3+4":

Notice that Theorist does not instantly calculate the answer. Theorist is not a pocket calculator; it has a bit more finesse. Theorist is waiting for you to tell it what to do. Who knows, maybe you want to type more into the equation.

Now, select what you just typed and choose **Calculate** from the **Manipulate** menu. Theorist now calculates the answer like a pocket calculator:



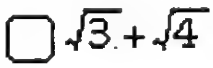
Select 3+4 again, but this time choose **Simplify**, which is right below **Calculate** on the **Manipulate** menu:



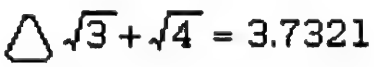
Theorist simplifies the 3 + 4 and turns it into 7. You might be thinking that Simplify and Calculate do the same thing. They don't.

Let's calculate something different. Select the 3 in the original expression, and click on the $\sqrt{x}$ symbol on the palette so that it looks like this:

Select the 4, and click on the $\sqrt{x}$ button again:

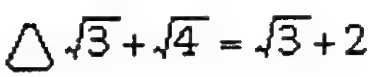


Now, select it all and Calculate:



Theorist calculates answers to nineteen digits of precision, but only five digits are shown in this case because our precision preference is set to five. If you pull down the **Prefs** menu and its **Display Precision** submenu, you can choose the precision you need.

Now, select it again and Simplify.



Simplify simplifies it as best it can. That means simplifying the square root of 4 to a 2, but that also means leaving the square root of 3 alone. Because the square root of 3 is unchanged, Theorist can't add the two numbers together, and leaves the equation unchanged.

Calculate and Simplify are different. Calculate gives numeric answers and Simplify gives answers in symbolic form.

Symbolic answers and numeric answers are useful at different times. For instance, the symbolic answer tells you that if you subtract 2 and square the number, you get exactly 3, to an infinite amount of precision. If you need to cut a piece of wood that length, though, you probably need to use the numeric form.

Variables

Theorist allows variables to represent unknown or unspecified values.

Type:

```
x+3*8
```

This gives you:

$$\square\ x+3\cdot 8$$

Now, try to Calculate it. You get this:

$$\square\ x+3\cdot 8 = x+24$$

Theorist doesn't know what value to use for x so it leaves it unspecified. Notice that no truisms have been violated, though.

You can have definitions of the variables used in calculations. Type in this formula:

```
x=5*7
```

This gives you:

$$\square\ x = 5\cdot 7$$

Go back and select the equation and choose **Make Working Stmt** from the **Notebook** menu. This tells Theorist to look here when it needs a value for x in a calculation, in the event you have more than one equation that could specify the value of x. Once you do this, a dot in the square reminds you that this is a working statement:

$$\boxdot\ x = 5\cdot 7$$

Now, if you go back and calculate $x + 3*8$, it gives an answer:

$$\triangle\ x+3\cdot 8 = 59$$

However, if you use Simplify, nothing happens. working statements only work for Calculate.

Numeric and symbolic variables are used extensively in later tutorials.

Functions

Variables are only one class of name. Another class is Functions. For instance, the name log is a function that takes the logarithm (base 10) of a number.

Press Return and type "`log(1000)`":

$$\square \log(1000)$$

When you Calculate it, you get:

$$\triangle \log(1000) = 3$$

Theorist has a wide variety of built-in functions, including natural log, trigonometric, and hyperbolic functions, which are explored further in later tutorials. For a complete list of built-in functions, look up "Predefined Names" in the Theorist Reference Manual.

To supplement Theorist's predefined functions, the Special Functions folder (in the Mathematics folder) that comes with Theorist contains more advanced functions.

Undefined Names

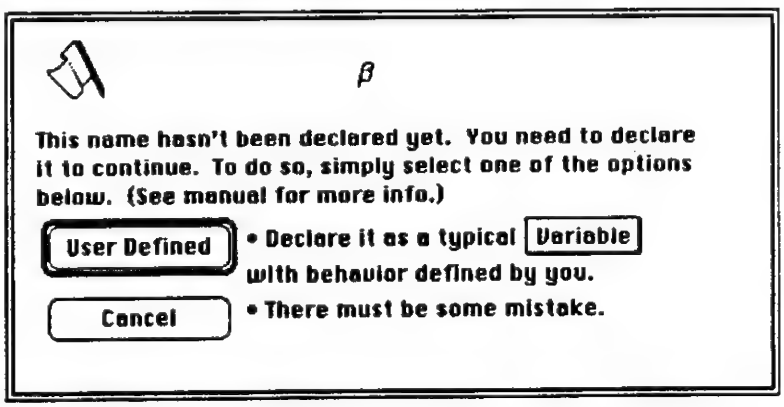
If you try to use a name that Theorist does not recognize, Theorist eventually notices. Theorist checks with you to make sure it is not some other name mistyped. If it is a new name, confirm it at this time.

The Option-B keystroke gives you the Greek letter beta.

For example, type:

Return Option-B

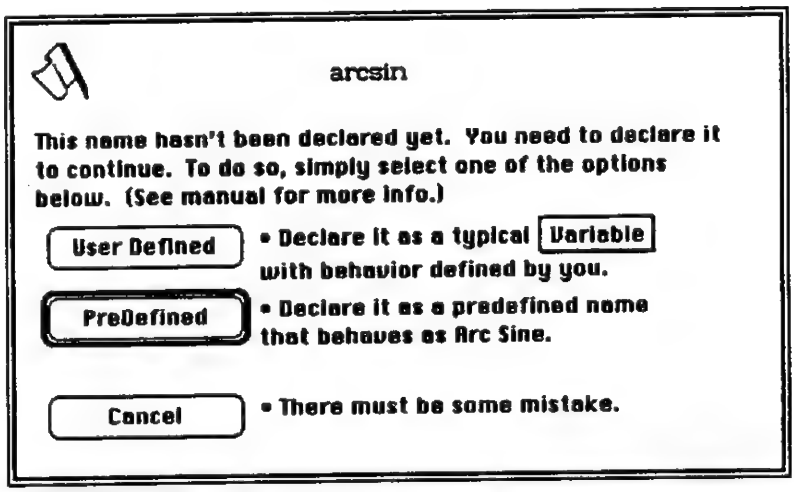
Select it, then choose **Clarify** from the **Notebook** menu. Theorist shows you this dialog:



Theorist asks you (the user) to define β. You can use the pop-up menu (**Variable** above) to set its class; this is explored in another tutorial.

Press Return and type the name "`arcsin`" and press Command-space to Clarify again. Since arcsin is a name that Theorist recognizes, you

get a different dialog:



Theorist recognizes the name arcsin because it is in Theorist's internal library, and assumes you want it predefined to behave as Arc Sine. However, it asks you to either verify this or define it as something else.

If you use a name that you don't intend to have a special, predefined behavior, click on **User Defined**. For example, enter:

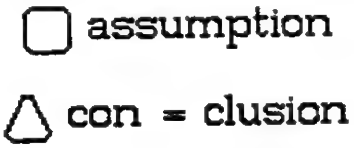
```
a=b+c+d+e
```

Now select it and Clarify. Theorist asks you to define *b*, *d*, and *e*. In the case of both *d* and *e*, there are special symbols from Calculus named *d* and *e*, and Theorist gives you a chance to use them that way. Since you want to use them as normal variables in this case, you want **User Defined**.

The Name Declarations tutorial explores this area further.

Propositions

You've probably noticed expressions have a square icon to their left, whereas the answers have triangular icons. Square icon expressions are called "Assumptions"; they are expressions or equations that you type in. Triangular icon equations are called "Conclusions". They are always equations; they are always the result of Theorist doing some sort of calculation or manipulation.



Theorist always generates an answer in the form of an equation, even if you start with just an expression. This initial expression is so that everyone remembers the question. This is important. Otherwise, at the end of a session, you would have little fragments of expressions all over the screen and it would be easy to forget what goes with what.

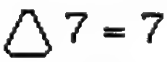
Theorist works with "propositions", which are statements that are assumed to be true. When you calculate an expression 3 + 4, Theorist makes an equation 3 + 4 = 7. When you give Theorist an expression, it converts it into an equation.

In some computer systems, the equal sign defines a name, moves a number to a storage place, or does something similar. For instance, you might have a statement $n = n + 1$. When this statement is executed, it takes the number in storage place n, adds one to it, and restores it in storage place n. In one of these systems, the left side of the equation must always be a variable; $n + 1 = n$ would be an error.

In Theorist, an equal sign means the same thing that it does in normal mathematics: it is a statement that two expressions on either side of the equal sign are the same, as in the statement 3 + 4 = 7. If you turn it around backwards, it means the same thing: 7 = 3 + 4. In Theorist, the equation $n = n + 1$ (which is equivalent to $n + 1 = n$) eventually leads to the contradictory equation 0 = 1. Theorist does not stop you from doing something like this; it depends upon you, the intelligent human being, to have the good judgment to determine that there is a mistake somewhere. (There are more complicated situations where only the human can judge what is a contradiction and what is insight; an equation such as $\sin(x) = 2$ could be a contradiction or could be a solution to a problem.)

Theorist is also non-procedural. This means the equations are not executed sequentially, they are all interpreted as a state of being, a set of facts that are assumed to be true. The order of the equations is almost irrelevant.

Theorist makes equations because if you think in terms of equations, it is harder to make a mistake. This becomes clearer later, but for now, try selecting the 3 + 4 on the left hand side of 3 + 4 = 7 and Calculate again:



This is an equation, which is true, based upon the last equation, which is also true. All Theorist algebra is done in a similar way, by generating equations that are true because they are based upon other equations that start out being true. By maintaining this framework, Theorist ensures the integrity of your derivation.

Automatic Conclusion Elimination

Select either 7 from the last equation, 7 = 7. Then, Calculate or Simplify. Theorist does not produce an answer. If you look closely, you will notice that an answer flashes briefly where an answer would go, but Theorist deletes it quickly as soon as it realizes that the answer is the same as the question. Since there is nothing new to tell you, Theorist saves space by deleting it.

Linear Algebra

Theorist can also do matrices. For instance, press Return and type `"(1,2;3,4)*(5,6;7,8)"`. (Be careful to type commas or semicolons where appropriate—they do different things.) This gives you:

$$\square\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}5&6\\7&8\end{pmatrix}$$

Now, select both matrices together and choose **Expand** from the **Manipulate** menu. (Simplify is not strong enough to do this.)

$$\triangle\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}5&6\\7&8\end{pmatrix}=\begin{pmatrix}19&22\\43&50\end{pmatrix}$$

Theorist is doing this symbolically, not numerically. Change the 2 in the first matrix to an x. Then select both matrices and Expand again:

$$\triangle\begin{pmatrix}1&x\\3&4\end{pmatrix}\begin{pmatrix}5&6\\7&8\end{pmatrix}=\begin{pmatrix}7x+5&8x+6\\43&50\end{pmatrix}$$

Theorist considers a scalar (a non-matrix) to be the same thing as a diagonal matrix. Type `"(1,2,3;4,5,6;7,8,9)+x"`, which gives you:

$$\square\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}+x$$

Drag over the expression (or double-click on the plus sign) to select it, then Expand it:

$$\triangle\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}+x=\begin{pmatrix}x+1&2&3\\4&x+5&6\\7&8&x+9\end{pmatrix}$$

It's as if the x turned into x times the identity matrix. You do not have to make up an identity matrix; Theorist simply does the right thing for situations such as this.

MANIPULATING EQUATIONS

In this tutorial you learn the nuts and bolts of equation manipulation in Theorist.

In a new notebook, type:

```
y=((3/7)*x-(9/4)+sqrt(2))^5
```

...which gives you on the screen:

$$\square y = \left(\left[\tfrac{3}{7}\right]x - \left[\tfrac{9}{4}\right] + \sqrt{[2]}\right)^{5}$$

Commute

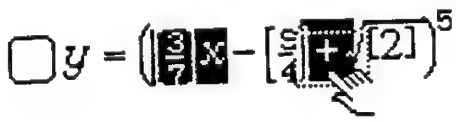
Theorist lets you do a great deal of the mundane manipulation by simply clicking and dragging. For example: in the above proposition, select the $\frac{3}{7}x$ so it looks like this:

$$\square y = \left(\left[\tfrac{3}{7}\right]x - \left[\tfrac{9}{4}\right] + \sqrt{[2]}\right)^{5}$$

Do not be alarmed when Theorist changes parentheses to brackets and braces. This has no effect on how the equation is manipulated; it only affects the screen display.

Hold down the Command or Option key (either or both). Notice how as you move the cursor over the selection, the cursor changes to a hand. With the Command key still held down, click down and drag the selection around the screen. You drag its outline, like dragging icons in the Finder. If you drag it along the equation, certain elements of the equation highlight, indicating that you can release the mouse button and drop your selection in there, but don't do it yet.

Drag your selection right, over the addition sign. When it highlights,



...release the mouse button. Theorist moves the $\frac{3}{7}x$ in between the $-\frac{9}{4}$ and the $\sqrt{2}$ so you get:

$$\triangle y = \left(-\frac{9}{4}+\frac{3}{7}x+\sqrt{2}\right)^5$$

Note: Theorist removes the unnecessary brackets after the manipulation.

You can commute anything in a sum in this way. Try it with the three expressions added together.

Note: Select a negated expression by clicking on the minus sign.

You can also commute factors in a product. Try doing it on the $\frac{3}{7}x$ expression. Command-drag the x to the left:

Rearranged, it becomes:

$$\triangle y = \left(-\frac{9}{4}+x\,\frac{3}{7}+\sqrt{2}\right)^5$$

Also try selecting the whole sum of three expressions and choose **Commute** from the **Manipulate** menu. This turns them all backward.

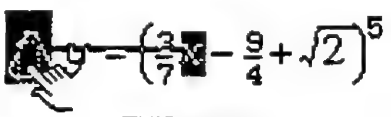
Exercise

Return the equation to its original form:

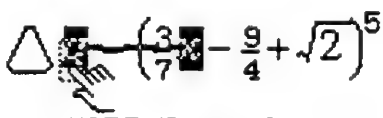
$$\triangle y = \left(\frac{3}{7}x-\frac{9}{4}+\sqrt{2}\right)^5$$

Isolate

The equation you have defines a relationship between two variables. There are many equivalent ways to see that relationship. One way is with x as a function of y. Select x and with Command pressed, drag it all the way past y over the proposition icon. When the icon highlights, a line goes across the equation from the icon to the x, indicating you can drop the x to isolate it:



THIS (Isolate)



NOT THIS (Move Over)

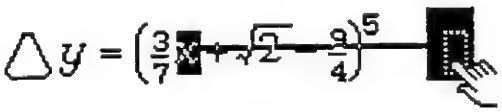
Go ahead and do it, which results in:

$$\triangle x = \frac{7}{3}\left(y^{\frac{1}{5}} - \sqrt{2} + \frac{9}{4}\right)$$

Now use the same process to isolate the y and get back to what you had before you started experimenting with isolating.

$$\triangle y = \left(\frac{3}{7}x + \sqrt{2} - \frac{9}{4}\right)^5$$

Next, try dragging the x all the way to the right instead of left.



...which is another way of isolating x.

$$\triangle \frac{7}{3}\left(y^{\frac{1}{5}} - \sqrt{2} + \frac{9}{4}\right) = x$$

Now try isolating some of the numbers. You can get some weird logarithms this way, such as:

$$\triangle 9 = 4\left(\frac{3}{7}x - y^{\frac{1}{5}} + \sqrt{2}\right)$$

$$\triangle 5 = \frac{1}{\log_y\left(\frac{3}{7}x + \sqrt{2} - \frac{9}{4}\right)}$$

Note: You're in for a hard time if you try to isolate y from $\log_y$. In this case, first isolate log from the argument.

Next, isolate y to get your equation back to its original form once again.

$$\triangle y = \left(\frac{3}{7}x + \sqrt{2} - \frac{9}{4}\right)^5$$

There is one last manipulation in this series that is somewhat different.

Move Over

Using the isolate manipulation, try to get $0=\left(\frac{3}{7}x+\sqrt{2}-\frac{9}{4}\right)^5-y$.
You can't! The only way is by taking:

$$\square\, y=\left(\frac{3}{7}x-\frac{9}{4}+\sqrt{2}\right)^5$$

...and moving the y to the other side of the equal sign,

...so you get:

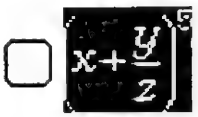
$$\triangle\, 0=\left(\frac{3}{7}x+\sqrt{2}-\frac{9}{4}\right)^5-y$$

Expand

Type the following:

```
(x+y/z)^5
```

Select the expression:



Choose **Expand** from the **Manipulate** menu or press Command-6, which multiplies out the terms and yields a longer expression.

$$\triangle\left(x+\frac{y}{z}\right)^5=\frac{y^5}{z^5}+5\frac{x^4y}{z}+5\frac{xy^4}{z^4}+10\frac{x^2y^3}{z^3}+10\frac{x^3y^2}{z^2}+x^5$$

Collect

In the above equation, select the first three terms to the right of the equal sign:

$$\triangle\left(x+\frac{y}{z}\right)^5=\frac{y^5}{z^5}+5\frac{x^4y}{z}+5\frac{xy^4}{z^4}+10\frac{x^2y^3}{z^3}+10\frac{x^3y^2}{z^2}+x^5$$

Choose **Collect** from the **Manipulate** menu or press Command-7, which collects factors common to all terms so they are multiplied onto the side:

$$\triangle\left(x+\frac{y}{z}\right)^5 = \frac{\left(y^4+5x^4z^4+5xy^3z\right)y}{z^5}+\cdots\cdots$$

The menu equivalents for these manipulations are similar, but not identical to the Command-drag manipulations. See the Theorist Reference Manual for details.

There are many other manipulations at your disposal, as you can see by looking at the **Manipulate** menu. These are explored in later tutorials and explained in the Theorist Reference Manual.

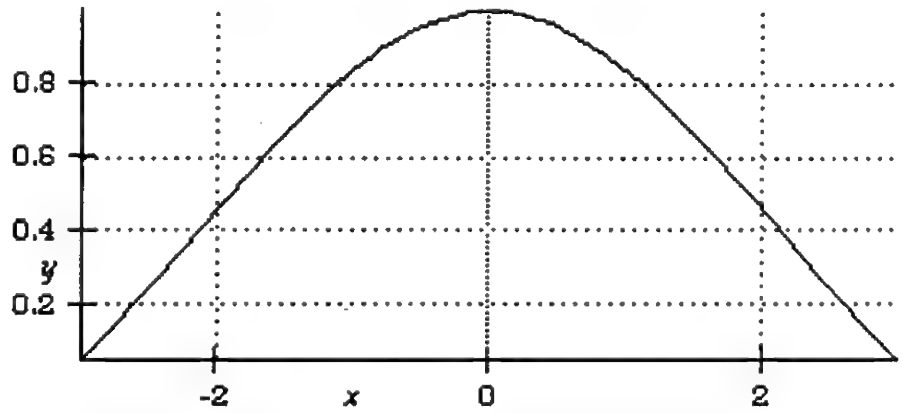
Graphing

2-D GRAPHING

In a new notebook, type:

```
y=sin(x)/x
```

With the cursor still in in the equation, pull down the **Graph** menu and from its **y = f(x)** submenu, choose **Linear** (or press Command-G):

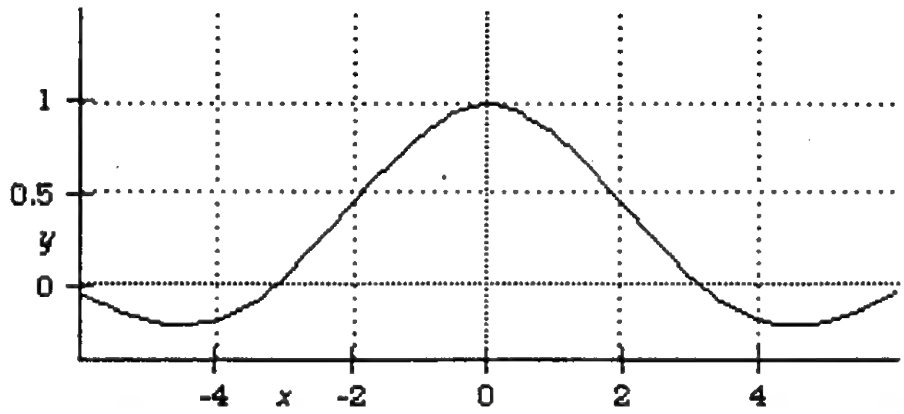


Hand Cursor

This only shows a small part of a very interesting graph. When you move the cursor over the graph surface, it turns into a hand. You can scroll the graph with the hand by dragging it. As you do, the axis labels change to indicate what part of the graph is visible.

Rocketship Button

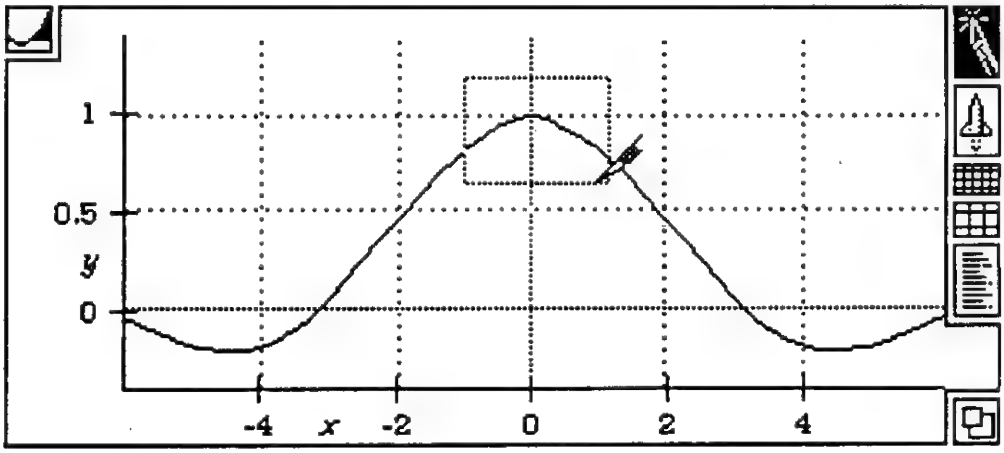
If you get frustrated crawling along a graph and want to get a more global view of it, simply click on the rocketship icon in the upper-right section of the graph window. It zooms your graph out by about a factor of two in each dimension.



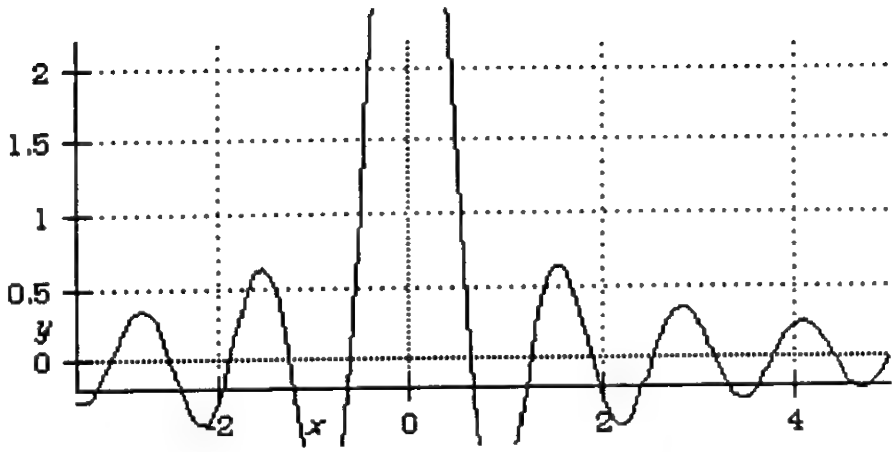
Knife Button

To zoom in on a specific section of the graph and see more detail, click on the knife icon. Your cursor turns into a knife, ready to slice out and enlarge a piece of the graph. Click on the graph surface itself and drag over a rectangle that you want to see closer. It is usually best to drag your rectangle so that it tightly encloses the part you want to see:

Tip: You can zoom in on a graph by holding the Option key as you click the rocketship.

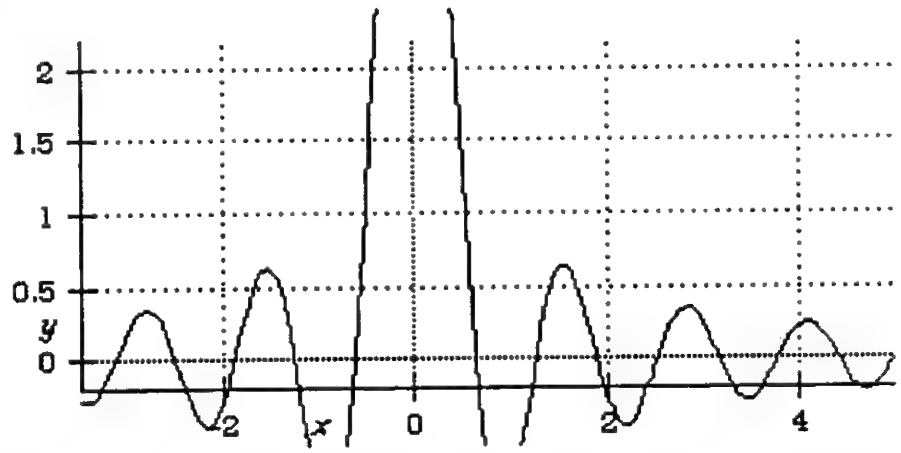


Theorist graphs are interactive and always reflect current conditions. When you change the equation, the graph changes in response. For instance, select the *x* from sin(*x*) and type "*5". Use a few clicks of the rocketship to zoom out.



More Accuracy Button

Some graphs may look chunky. The default graph has a small number of data points, for greater speed. For more points, and a higher resolution plot, click on the more accuracy button on the graph window. (This has a more noticeable effect on other graphs; this particular graph does not change appreciably.) As you click the button, notice that two numbers appears next to it. The left number is the current resolution, and the right number will be the new resolution when the graph is redrawn.



Dragging size box

The size box in the bottom-right corner of the graph window can be used to resize the graph. Drag it outward to see a bigger graph.

Exercise

Make a 2-D plot of $y = \sin x$ and note the numbers on the axis labels. Zoom out a couple times, then try to zoom back in by using the knife to cut out a portion in the original dimensions. See how close you can get with practice.

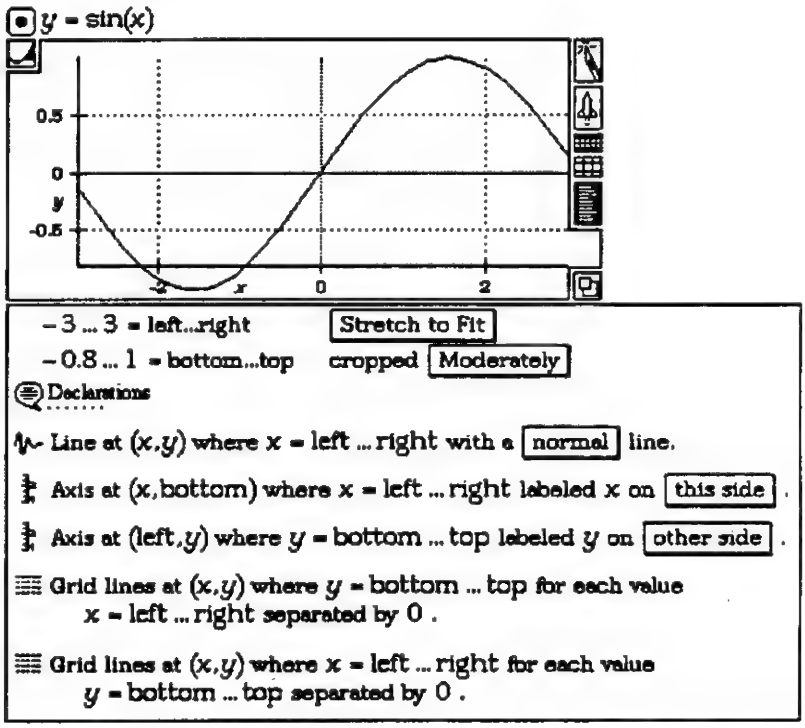
INSIDE GRAPHS

A Theorist graph is a space for drawing graphic objects. A plot is a graphic object drawn inside a graph. In this example, we explore a graph and its plots.

Type "`y=sin(x)`" into a new notebook.

Graph Details Button

Create a graph by pressing Command-G then click on the graph details button to see details about the graph.



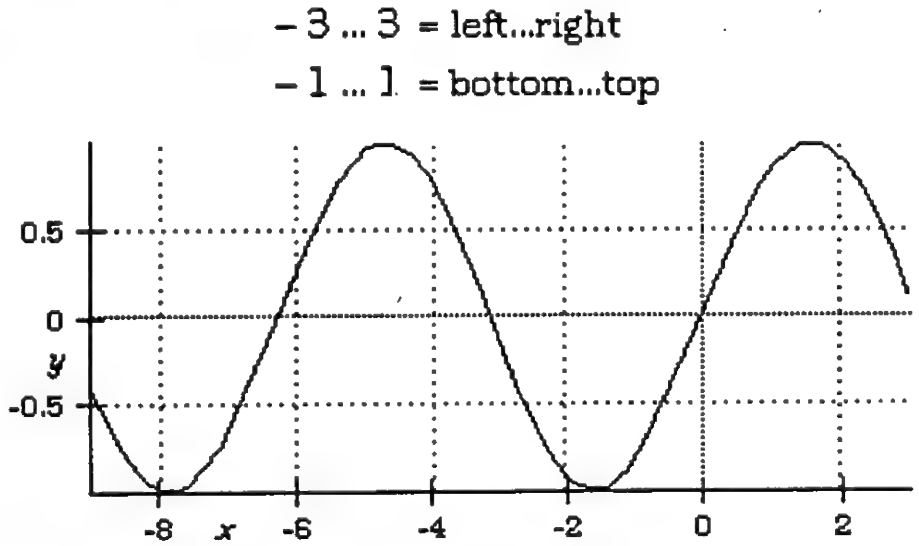
Locate the first two lines in the graph details:

$$-3 \ldots 3 = \text{left...right}$$

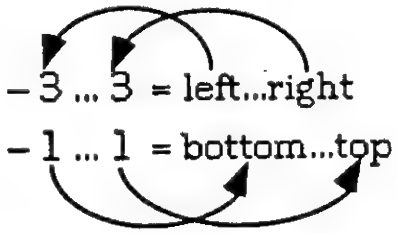
$$-1 \ldots 1 = \text{bottom...top}$$

These define the graph bounds, or the range of x and y values visible within the viewport. Look at the x-axis on the graph and note that it ranges from –3 to 3. (Although it is only labeled at –2, 0, and +2, you can see where –3 and +3 are).

Change the –3 (which corresponds to left) to –9. You needn't disturb the minus sign. The graph redraws itself, this time showing more real estate on the left side, all the way to –9.



Note that the names imply their purpose.



Similarly, you can change the other three and manipulate them as desired. Note that when you use the knife or rocketship you are actually causing Theorist to intelligently modify these values and, subsequently, the graph.

Note the elements in the image of the graph. (See below.) There are vertical and horizontal axes, a sine wave, and crosshatched grid lines. Each is independent and can be manipulated individually, allowing you to control the appearance of the graph.

Below the graph, in the details, you can see the individual plots that allow you to control the graph. Each icon starts a new plot. There are two axis plots, one line plot, and two grid plots (one for vertical lines and one for horizontal). Manipulating the plots allows you to control what is drawn.

Line at (x,y) where x = left ... right with a [normal] line.

Axis at $(x,$bottom$)$ where x = left ... right labeled x on [this side] .

Axis at $($left$,y)$ where y = bottom ... top labeled y on [other side] .

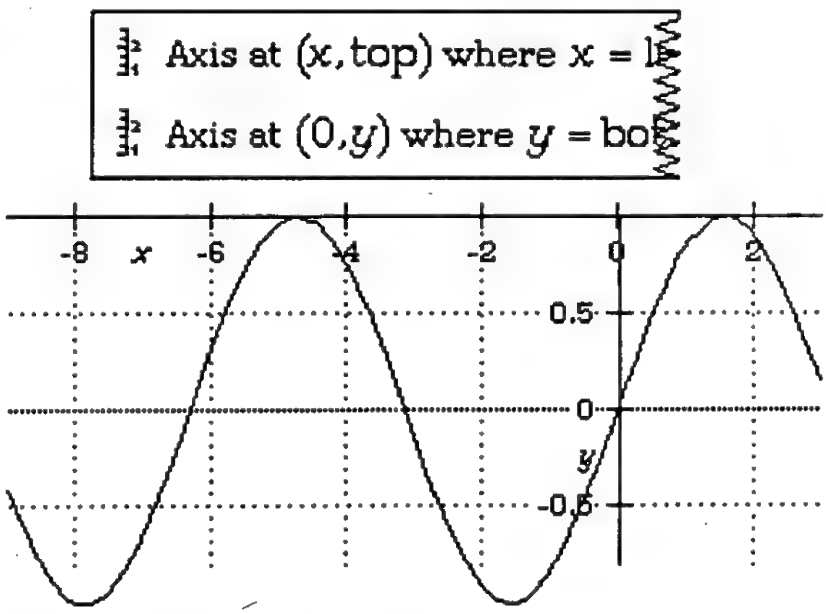
Grid lines at (x,y) where y = bottom ... top for each value x = left ... right separated by 0 .

Grid lines at (x,y) where x = left ... right for each value y = bottom ... top separated by 0 .

Axes

Axis Icon

You can move the bottom horizontal axis to the top. Find the axis plot that starts out "x, bottom". Change the word "bottom" to "top". Use the same technique to change the vertical axis from the left side to the right. You can also move the vertical axis to zero (or another value if you want) by typing "0" in place of "left" or "right":



You can move the axis to any numerical value by typing in that value.

Grids

You can remove a set of grid lines by clicking on its icon to select it, then pressing Delete. Note that one set of grid lines defines the vertical lines and the other horizontal lines. Delete both to see the graph without grids.

Grid Icon

You can similarly remove the axes by clicking on their icons and pressing Delete. Do this now and the graph redraws with no grids and no axes.

Line Plots

Line Plot Icon

You can delete the line plot itself. Do this now and notice that the graph has nothing drawn in it. Since there is nothing left in this graph, delete it by clicking on its icon and pressing Delete. Select some part of your equation and press Command-G to make another graph with everything in it.

Now you can learn in detail how a plot works. Open up the graph details and locate the line plot:

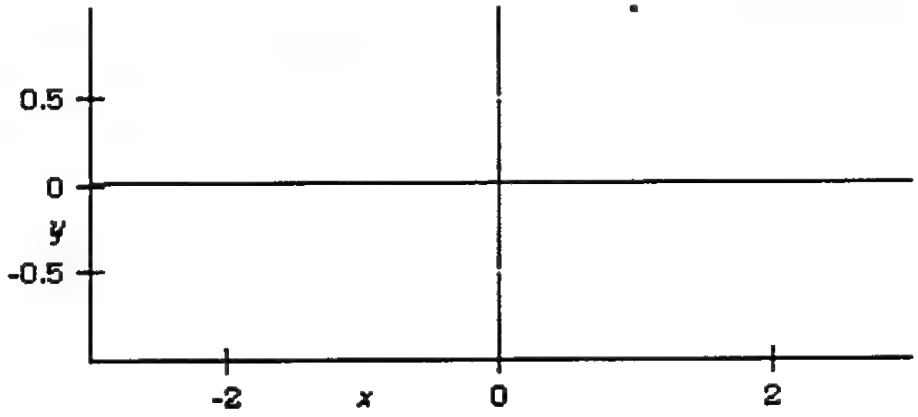
Line at (*x*,*y*) where *x* = left ... right with a [normal] line.

The last part is a pop-up menu. You can change the linestyle to any one of four options. This is useful if you have more than one line plot in a graph. Change it to dashed, dotted, then heavy and look at the results after each change. Leave it at heavy.

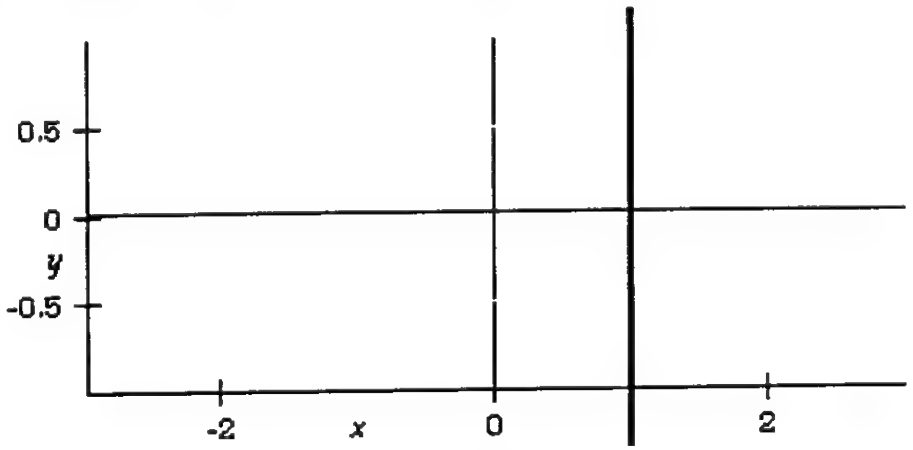
The first part, (*x*,*y*), is called the Vector. This is an expression that must, as the name implies, be a 2-vector. If this were a 3-D graph, it would have to be a 3-vector. 2-vector simply means it is two expressions with a comma in between.

For education's sake, you can change this vector to something very simple. Change the vector to (1,1) by selecting the whole vector and typing in the coordinates.

Notice that your graph turns into a single dot at (1,1). As you can see, the vector is a very important part of the line plot.



Next, change the second 1 to *x*. The graph turns into a vertical line.



The first coordinate of this line is always 1, while the second coordinate ranges from –3 to +3, as the second part of the line plot implies. It's just that the names are confusing.

You are not limited to *x* and *y*; you can use any variables.

Theorist is designed to draw a wide variety of plots, some of which are plots of *y* as a function of *x*. Some are plots of a variable named *P* as a function of *T*. Some are even plots of *x* as a function of *y*. As such, the vertical coordinate is not necessarily *y*, and horizontal coordinate is not always *x*. You can use any variables for these purposes.

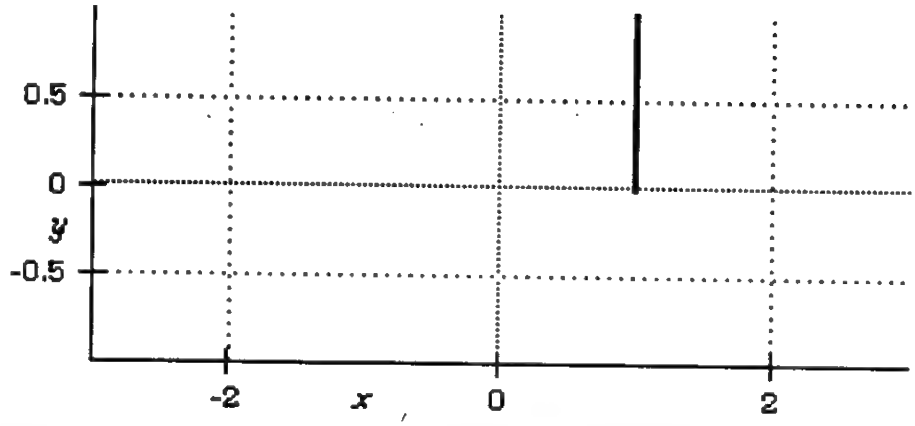
In Theorist, the rule is that the first coordinate is the horizontal one and the second the vertical. You can see this both in the order of the vector and in the order of the bounds at the top of the graph.

As you can see, the second part of the line plot, "*x* = left...right" is starting to become significant.

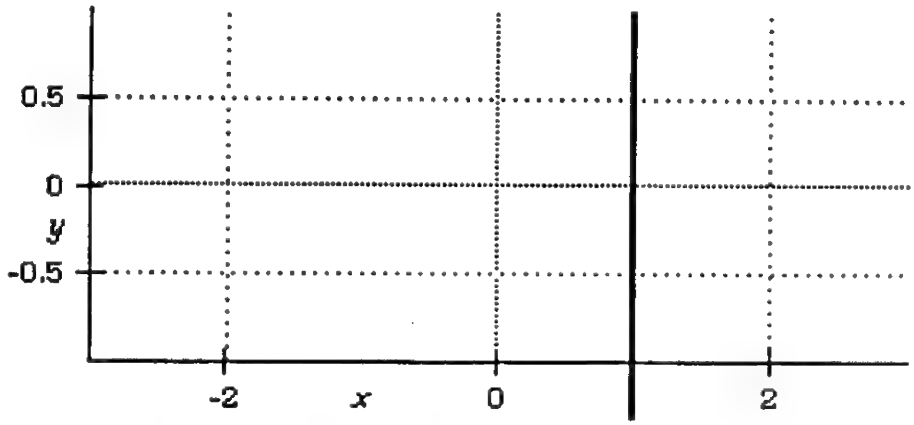
Remember we are still referring to the line plot.

The words left and right are simply names that are only allowed inside of a graph. They represent the graph bounds at the top of the graph. At present, the right represents 3 and the left –3. Change this to read "*x* = 0...1". (Type periods for the "..." part. If you have trouble, type "0.0" for 0 to make things perfectly clear to Theorist.)

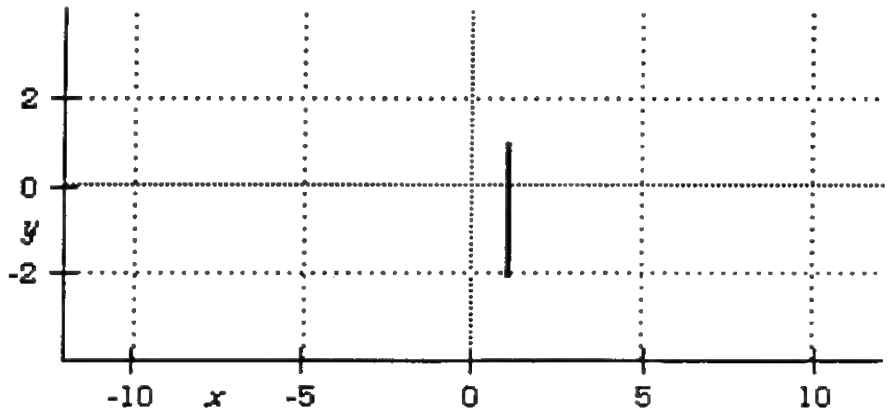
Your graph now looks like this:



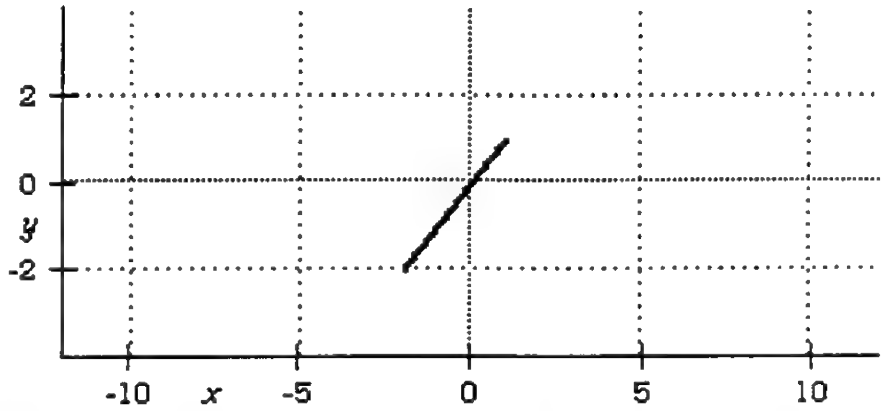
Change it to "$x = -2 \ldots 1$", and your graph looks like this:



You may want to zoom out to see the endpoints:



Now change the vector from $(1, x)$ to (x, x). The graph looks like this:



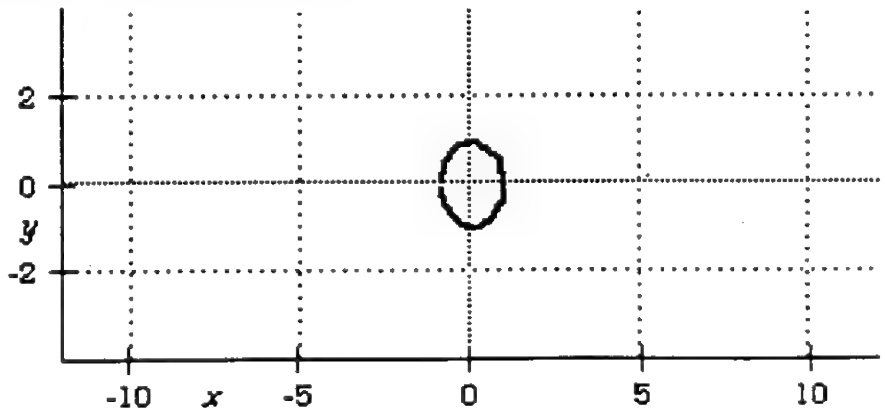
Now you can see how this line plot works. To find a point to plot, Theorist does a Calculate on the vector. To get successive points to make a line, Theorist defines x to have values ranging from –2 to 1.

In other words, the line plot is a miniature parametric plot. Change it to the following:

↬ Line at $(\cos[x], \sin[x])$ where $x = -3 \ldots 3$ with a heavy line.

You get an incomplete circle.

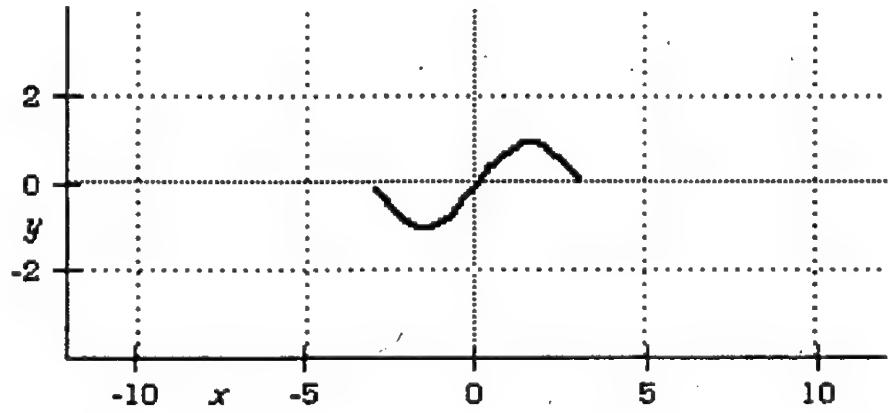
If the range is $-\pi$ to π, the two ends meet.



Now change the plot to read:

↬ Line at $(x, \sin[x])$ where $x = -3 \ldots 3$ with a heavy line.

You can see that this is sort of an unraveling of the plot that looks like your original graph.



In fact, since *y* is defined to be sin *x*, you can change that to read line plot at (*x*, *y*) and it is the same. That's what Theorist creates when you make a new graph.

GRAPHING PREPARED DATA

In this tutorial, you will see how to extract data from a graph, and how to make a graph from prepared data.

Type "`A=(`". Type "`;`" twenty-five times. Use **Select Out** from the **Edit** menu (Command-A) or drag until the whole vertical vector is selected. Choose **Select In** from the **Edit** menu or press Command-E. At this point, you should have twenty-five individual blinking selections.

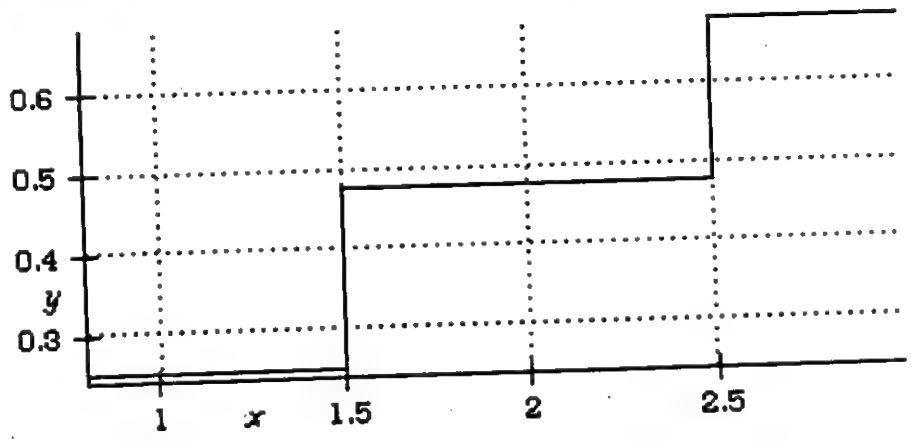
Type "`sin(`", which appears at each of the twenty-five insertion points, and then choose **Row Number** from the **Input** menu. At this point, make sure the last row is number twenty-five; if it is not, start over.

When Theorist asks you to declare A, make it an M-Linear op.

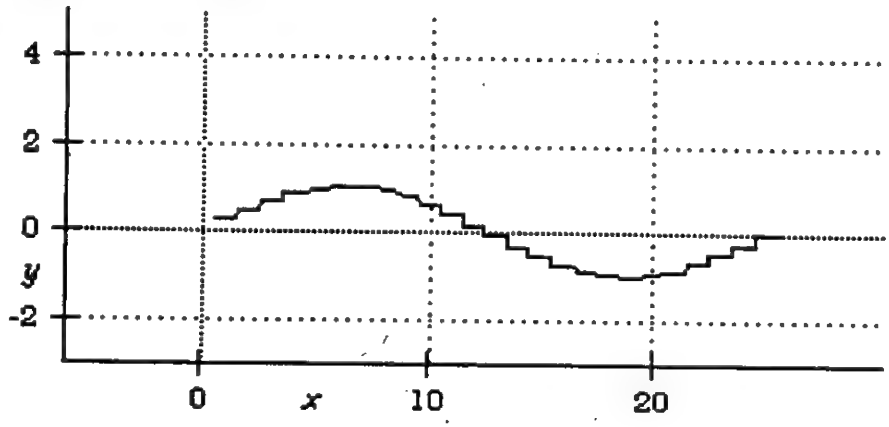
Type "`/4`" and then select each individual row by using **Select Out** three times. As you hold down *both* the Command *and* Option keys, choose **Calculate** from the **Manipulate** menu. (This does the calculation in place without creating a new proposition.)

Of course, if this were a real-life situation, you would probably have a spreadsheet or data acquisition program from which you would copy and paste data.

Press Return and type "`y=A_x`", then press Command-G. The graph looks like this:

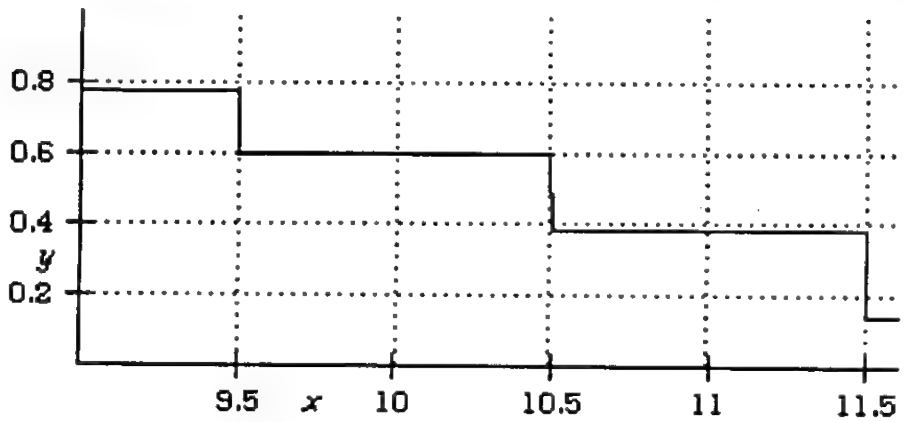


It looks stepped because the horizontal bounds are only one to three. Click on the rocketship icon four times, then use the hand icon to center the graph if necessary.



In our case, the number of points is 25, so we want our horizontal (x) bounds to be 1 through 25. If you have 50 data points, you want your x bounds to be 1 through 50. In other words, the x value is the index into the vector.

Zoomed out far enough (with the aid of the rocketship), the graph may appear to be smooth, but if you zoom in you can see what Theorist is doing. Take the knife and drag over a small portion of the graph. Note the stepping again.



Also note that it makes the transition between numbers (e.g., at 9.5, 10.5, 11.5, etc).
When Theorist calculates the value of A_x, it first calculates the value of x, then rounds it off to the nearest integer. If x is between 14.5 and 15.5, for instance, it arrives at the number 15. Then, Theorist takes the fifteenth element of the matrix A and uses that as the calculated result. This is why you get stair-stepping.

PARAMETRIC DATA GRAPH

This is a follow-up to the previous tutorial, Graphing Prepared Data, in which you draw a graph of y versus x, where x is the index into a long vector and y is the value at that vector. In this tutorial, we draw y versus x again, but this time, both y and x are taken out of a table or vector that you paste into Theorist.

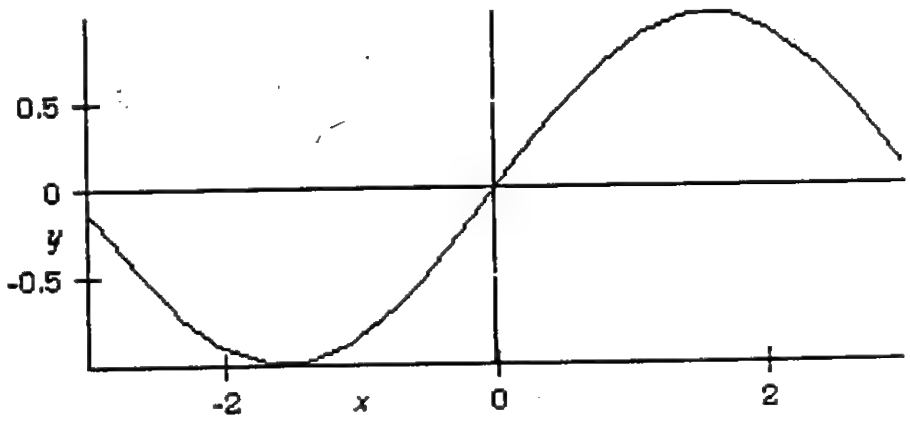
-3	-0.141
-2.652	-0.47
-2.479	-0.6156
-2.217	-0.7982
-2.043	-0.8903
-1.826	-0.9677
-1.696	-0.9923
-1.435	-0.9907
-1.304	-0.9648
-1.087	-0.8853
-0.9133	-0.7914
-0.6519	-0.6069
-0.4783	-0.4603
0.4783	0.4601
0.6519	0.6069
0.9133	0.7913
1.087	0.8853
1.222	0.94
1.304	0.9646
1.353	0.9765
1.435	0.9908
1.532	0.9993
1.696	0.9923
2.043	0.8904
2.217	0.7983
2.479	0.6157
2.652	0.47
3	0.1411

Notice the graph has more data points where it is more curved.

In this tutorial, we generate data for our graph from an existing graph. You can copy data out of a Theorist graph and paste it into a spreadsheet or other program. Conversely, you can copy data out of a spreadsheet and paste it into a Theorist matrix. We will put the two together and copy data out of a graph into a Theorist matrix.

Type "`y=sin(x)`", graph it, and select the graph. From the **Graph** menu, choose **Copy Points List**. Press Shift-Return for a comment and perform a paste. You get a list of pairs of numbers similar to those at the left of this page.

These are the actual x and y values Theorist uses to draw the graph of $y = \sin(x)$. They are formatted the way data is formatted when you copy it out of a spreadsheet: numbers are separated by Tab characters, and a Return character appears at the end of each line. You may notice the x values (the first column) are not evenly spaced between –3 and 3. To better draw the graph, Theorist generates more data points where the graph is more curved.



Now delete the graph and equation; you do not need them anymore. Be sure to use the Delete key or **Clear** menu item, *not* Cut, or you lose the data you previously copied to the clipboard. Delete your comment too if you wish.

Don't forget the comma.

You need a home for the two columns of numbers in the clipboard before you can do anything with them. Press Return and type "`A=,`" and you get:

$$\square\, A = (\blacksquare, ?)$$

Drag over the two question marks (or press Command-A to select out) and perform a paste. Theorist automatically resizes the matrix to fit the data. Now you have a matrix that looks like the table of data. (If Theorist asks you to declare *A*, make it an M-Linear op.)

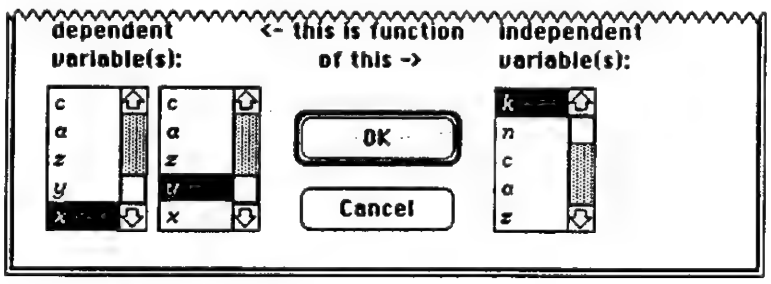
Next, press Return and type "`x=A_k,1`". Press Return again and type "`y=A_k,2`".

$$\square\, x = A(k, 1)$$
$$\square\, y = A(k, 2)$$

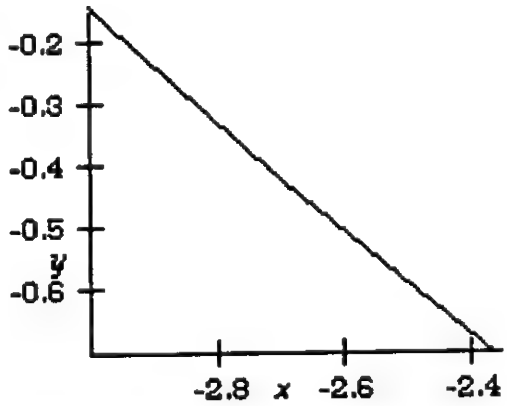
Now select *x*, or anything from the *x* equation. Then hold down Shift as you select *y* or anything from the *y* equation, so that parts of both equations are selected:

$$\square\, x = A(k, 1)$$
$$\square\, y = A(k, 2)$$

Choose the first item on the **Graph** menu, **x = f(t), y = g(t)**, which makes a parametric plot. Both the *x* and the *y* values are taken out of the table. (If Theorist gives you a dialog because it needs more information, make *x* and *y* the independent variables—in that order—and make *k* the dependent variable, as shown in the following screen shot of the dialog box.)



The graph appears as:



Unfortunately, as it stands the graph does not show the whole table of data because the parameter, *k*, ranges over a default range. When Theorist makes a parametric graph, it does not know what range to use for the parameter unless you define working statements. Now we will fix the graph. Open the graph details and locate the line plot:

∿ Line at (x,y) where $x = -3 \ldots 3$ with a [normal] line.

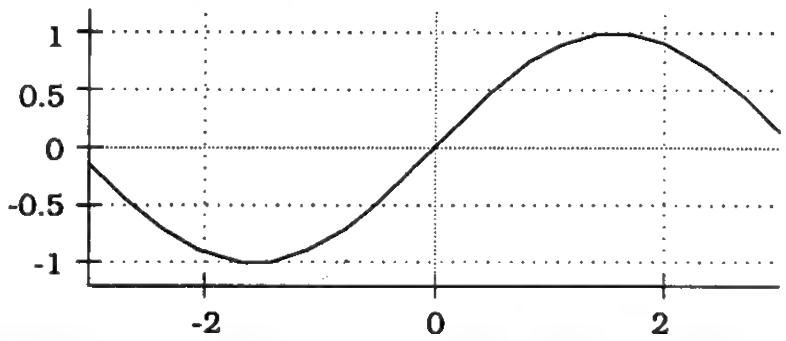
...and change the $k = -3 \ldots 3$ to $k = 1 \ldots$RowsOf(*A*).

∿ Line at (x,y) where $x = 1 \ldots$ RowsOf(*A*) with a [normal] line.

This is how you make a parameter range over all of the values of a matrix. RowsOf is a predefined function which gives the number of rows of the enclosed matrix. The first row is always numbered 1 and the last one is always the same as the number of rows.

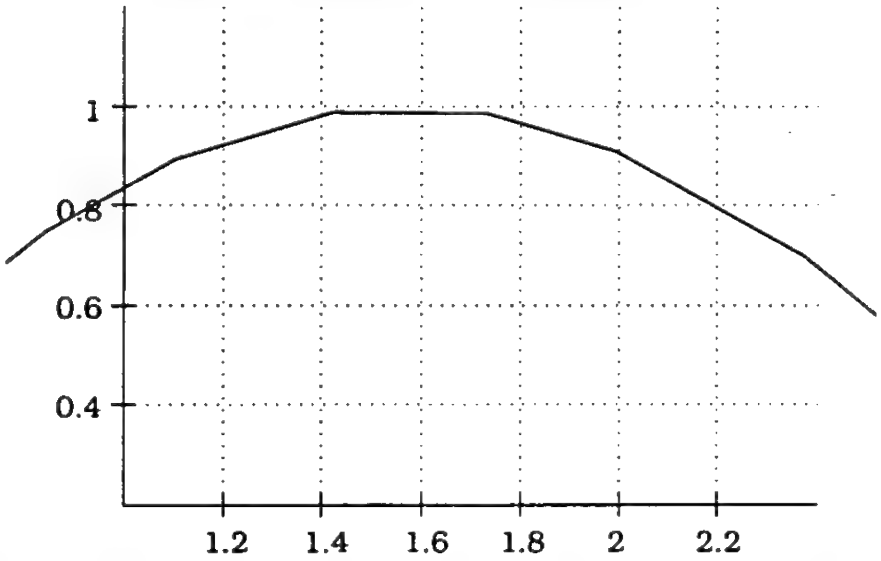
Close the graph details by clicking on the button again. Press Command-space to clarify and declare RowsOf as predefined when the dialog comes up.

You still have to adjust the graph bounds. You can do this with the knife and rocketship. But first resize the viewport larger. Then zoom out about four times. You recall that the x data points range from –3 to 3 and the y data points range from –1 to 1, so use the knife to zoom in on that particular area of the graph.



It looks right this time without uneven data points or steps.

Zoom in on the upper curved section of the graph.



If you zoom in close enough, you can make out the individual straight line segments which make up the curve. No matter how close you zoom in, though, you do not get a graph of sin(x) any more accurate than the original. In fact, you do not get any data points that are different from what is in the original data set, which is what Theorist drew in the original graph.

DESIGNING FUNCTIONS

One of the more amusing things you can do with Theorist is draw graphs of functions. Crucial to this amusement is the ability to engineer a function to look a certain way, a skill that goes beyond amusement and is important for modeling physical phenomenon.

Distorting the Output

In this tutorial you learn about functions and how you can distort them to fit your needs. The example function is sin(*x*).

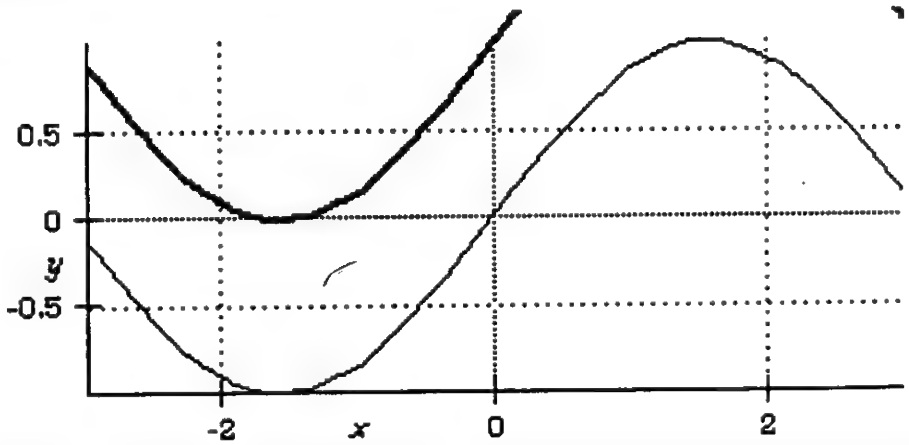
Start by typing:

```
y=sin(x)
```

Then type Command-G to graph it. A two-dimensional graph appears below your equation. Press Return and type:

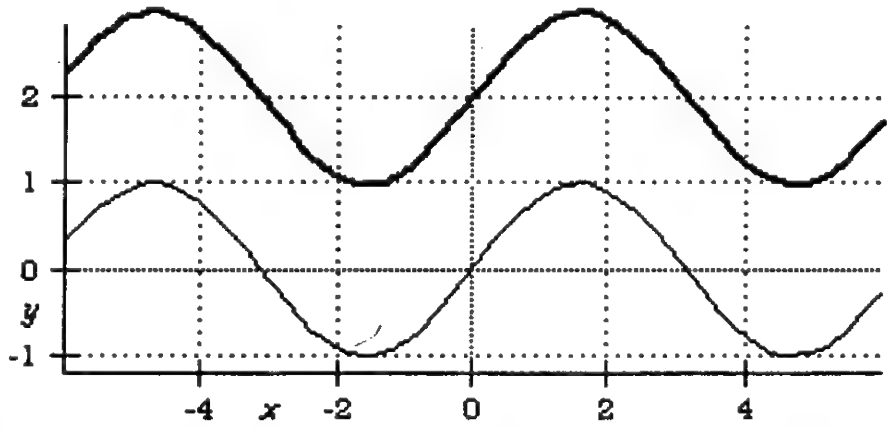
```
y'=sin(x)+1
```

Then pull down the **Graph** menu and choose **Add Line Plot**. (If you get any declaration dialogs from Theorist, press Return for each to accept its defaults.) Notice that another, thicker line appears on the 2-D graph.

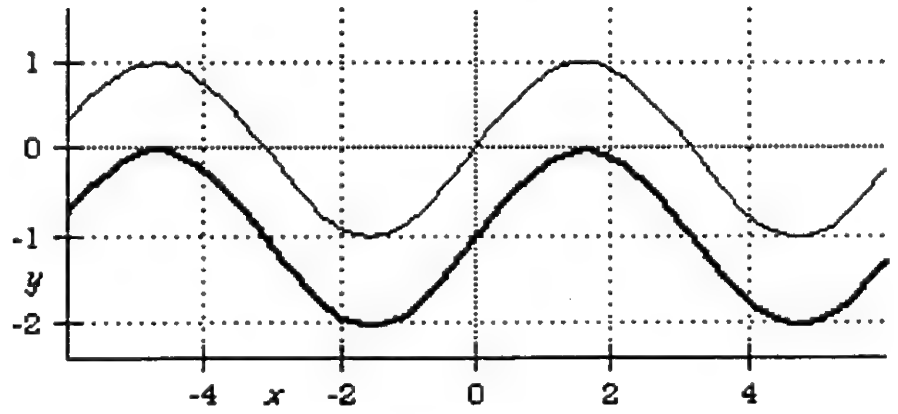


The thin line is our original sin(*x*). The heavy line is the same thing, modified with the +1. Notice that the thick line is—you guessed it—one unit above the thin line.

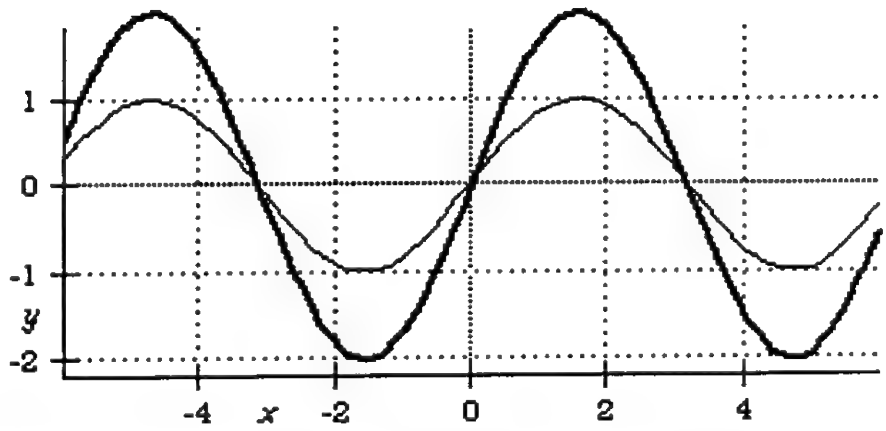
Change the 1 to a 2 by selecting the 1 and typing "2"; the graph redraws. You may have to zoom out with the rocketship icon and you may want to change the view with the hand icon.



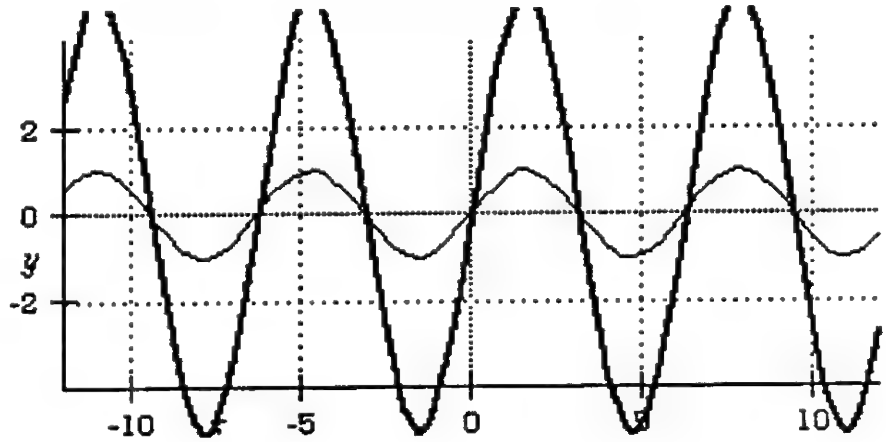
Now change the 2 to –1 by selecting the 2, pressing Delete, and typing "–1". The graph lowers by one unit. (Again, you may need adjust the view.)



Now get rid of the –1. (If you get something extraneous like "+?", delete it, too.) Select the sin(*x*) and type "*2". The sine wave now swings twice as far because the *2 amplifies the sine wave by a factor of 2.

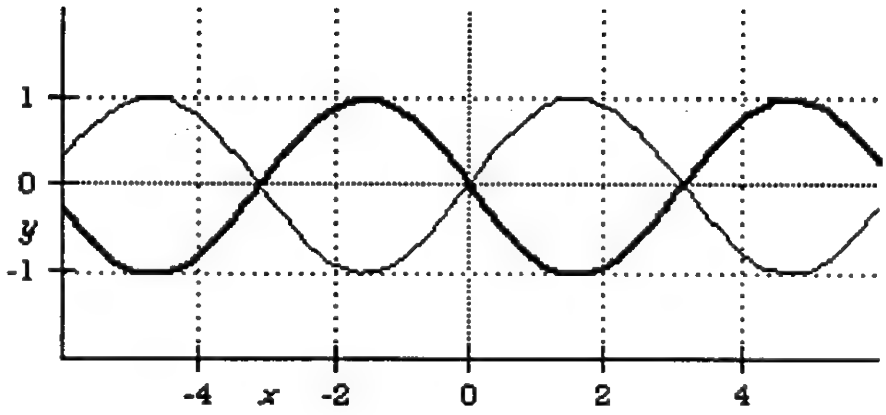


Change 2 to 5 and the effect intensifies even more. (Zoom out for a better view.)



Option-clicking the rocketship zooms back in.

Get rid of the *5 and change sin(*x*) to –sin(*x*). (Select sin(*x*) and press Option-minus to negate it.) This turns the graph upside-down. (Option-click the rocketship icon to zoom back.)

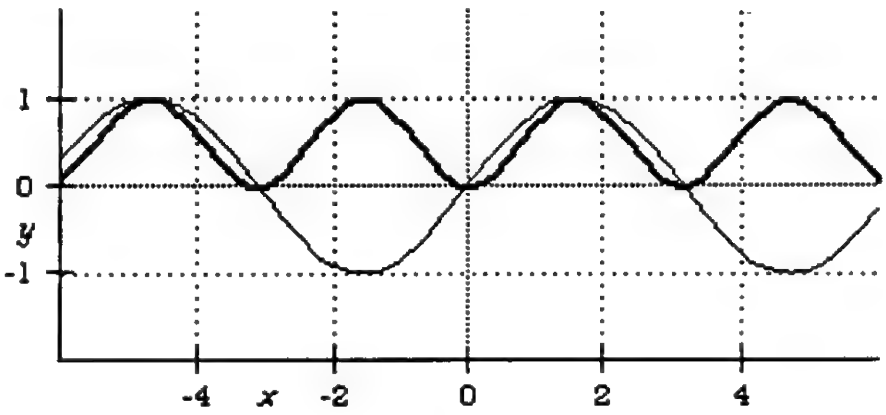


All of these modifications so far are linear; you could accomplish the same thing by drawing the graph on a rubberized sheet of paper and stretching it vertically or turning it upside-down. Now try some non-linear transformations.

Delete the –sin(*x*) and type:

```
(sin(x))^2
```

If necessary, press Command-space to redraw the graph.



Here are the effects of squaring:

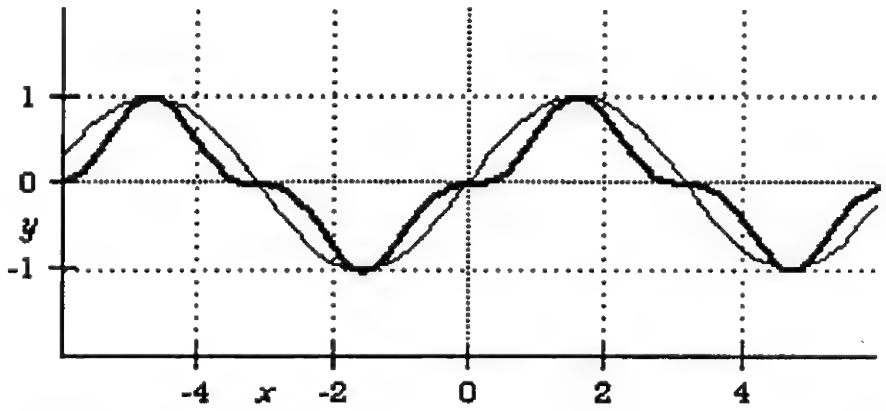
- the square is always positive
- if the value is less than 1, the square is even smaller
- if the value is greater than 1, the square is even larger (not observed here)

The result is that all the negative values are now positive and all of the places that zoomed through zero now form rounded curves as they bounce up again. The whole thing looks just like another sine wave. In fact, it is! There is a well-known trig identity which describes it.

Exercise

By looking at the graph of $(\sin[x])^2$, deduce the identity without using any algebra and without cheating and looking it up.

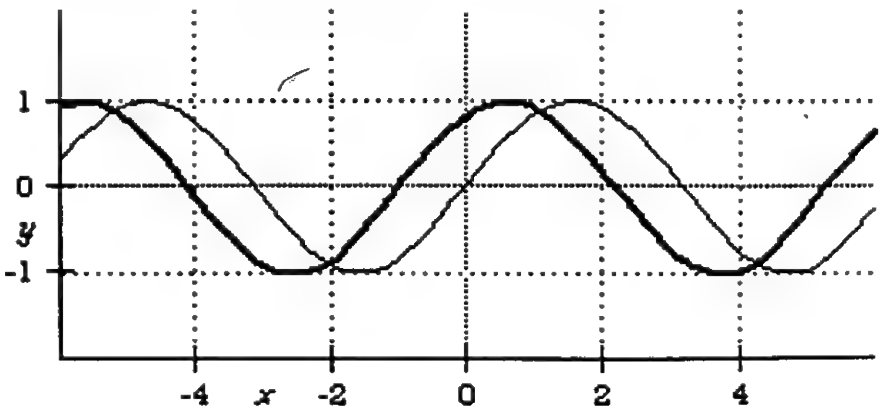
Now change $(\sin[x])^2$ to $(\sin[x])^3$. This is a very nonlinear transformation!



Distorting the Input

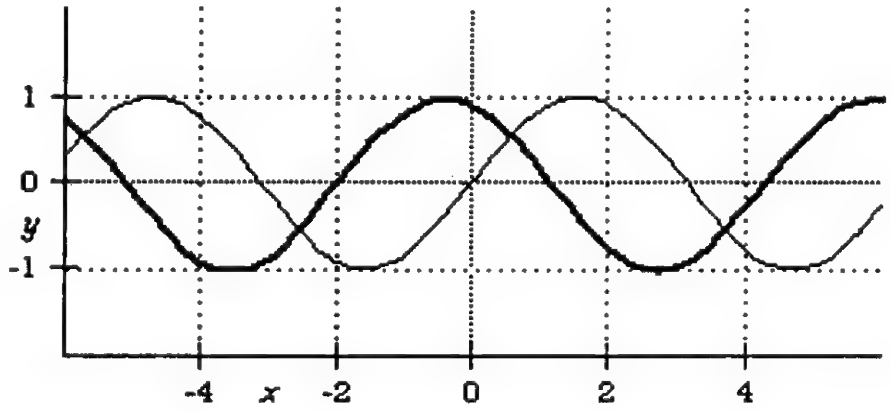
We just saw what happened when you distort the result of a function. Now we will see what happens when we distort the input. Not only will the effects be horizontal rather than vertical, but the effects will also be the reverse of what you would expect if the same distortion were applied to the output. Change $(\sin[x])^3$ to `"sin(x+1)"`.

That's sin(*x*+1), not sin(*x*)+1.

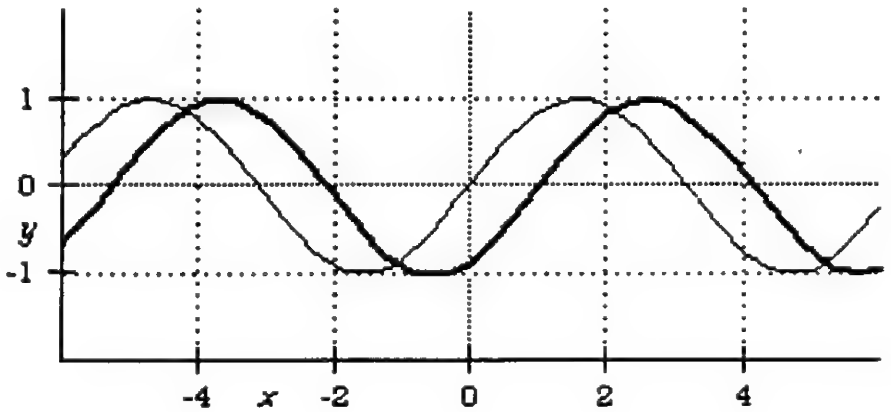


You can see that the +1 has the effect of moving the function over by one. But notice that it is in the the opposite direction from what you might think—it was moves one unit *backwards*. Think of it this way: in order to get sin(0), where the function crosses zero, you have to make x be −1 to cancel out the +1.

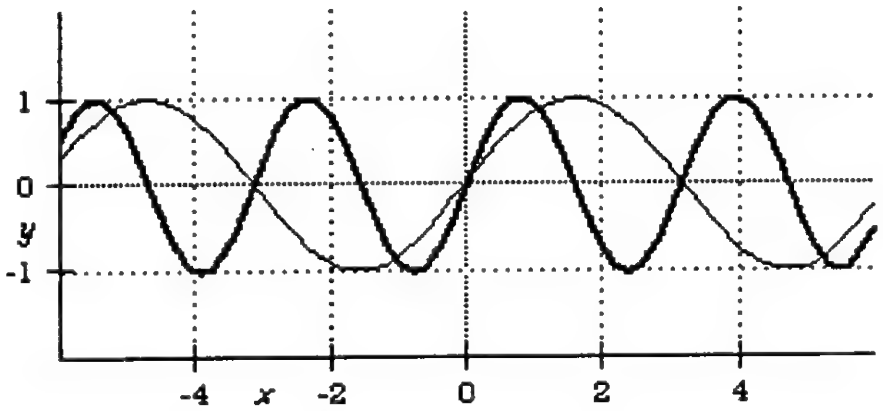
Now select the 1 and change it to 2. This moves it over the way you expect.



Now change the 2 to −1 by selecting the 2, pressing Delete, and typing "-1". Now the curve moves forward. Remember that in order to move it from zero to some place n, you must subtract n from x.

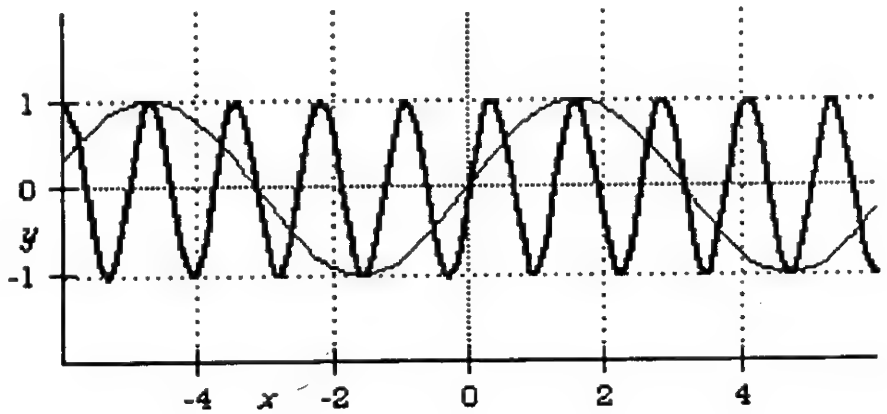


Delete the –1 and any extraneous ?, then, with *x* selected, type "*2".

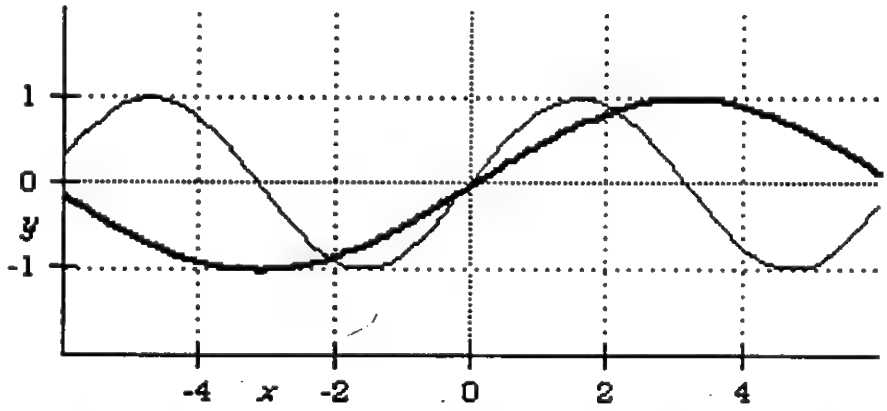


Notice that the higher frequency graph has more humps. Perhaps you expected the graph to expand out by a factor of two instead of compressing. Well, sin() is the same—you are just making *x* change twice as fast.

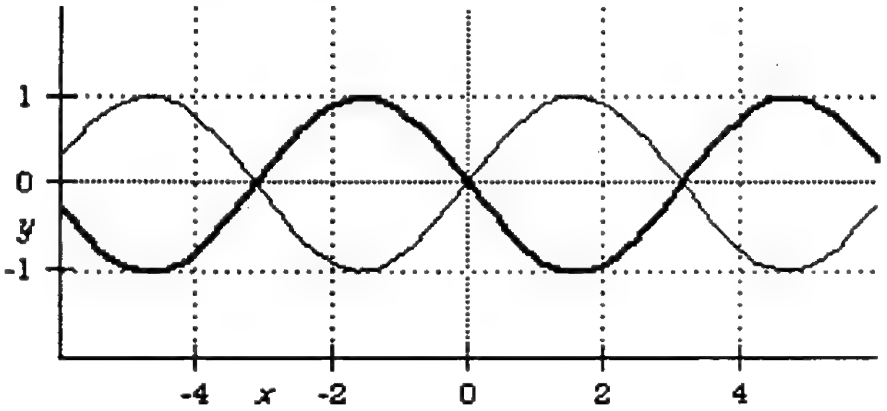
Now change the 2 to 5. Note that the effect intensifies:



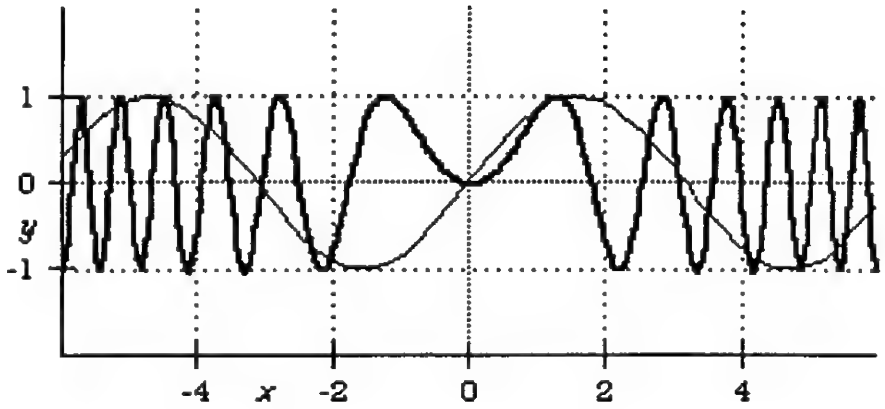
Change the 5 to $\frac{1}{2}$ (select 5 and type "1/2") and notice what it looks like:



Now Delete the $\frac{1}{2}$ and any extraneous ?, then, with *x* selected, type Option-minus to negate it, so the expression appears as sin(–*x*). Note that the function is turned around backwards:



Delete $-x$ and change it to x^2.



You can see the effect of x^2 in the behavior near $x = 0$. It starts off very slowly, in the positive direction, whether x is positive or negative. After a while, though, x accelerates and you get a wildly varying function. It is almost as if you have $\sin(5x)$ except the 5 is not 5, it is x, getting bigger and bigger as x gets larger.

GRAPHING IMPLICIT EQUATIONS

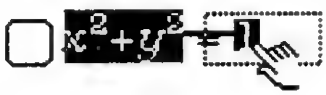
Many relationships between two variables are stated not as explicit functions, where one of the variables is a function of the other, but as implicit functions, where the variables are tangled up together in a seemingly hopeless algebraic mess.

For example, in the relationship $y = x^2 - 1$, $x^2 - 1$ is an explicit function. It is single-valued: for any given value x, there is no more than one value of y.

The relationship $x^2 + y^2 = 1$ is implicit, though. You cannot change it to the form $y = f(x)$ without doing a bit of cheating. If you try with Theorist's Auto Casing option on, Theorist splits the answer into two cases, $y = \sqrt{1 - x^2}$ and $y = -\sqrt{1 - x^2}$. If you try to graph them, each equation will give you a half-circle. The original relationship, however, defines a full circle. For every value of x, there are two values of y (unless x is –1 or less, or 1 or greater). A "function" cannot return both of these values at the same time. This is a difficult situation.

Fortunately, Theorist has a way to draw graphs of relationships of this nature. Type "`x^2 Esc +y^2 Esc =1`".

Next, select the left hand side by double-clicking on the plus sign. Hold down the Command key and drag the selection to the other side of the equation until it highlights,



...and then let go. You get:

$$0 = -x^2 - y^2 + 1$$

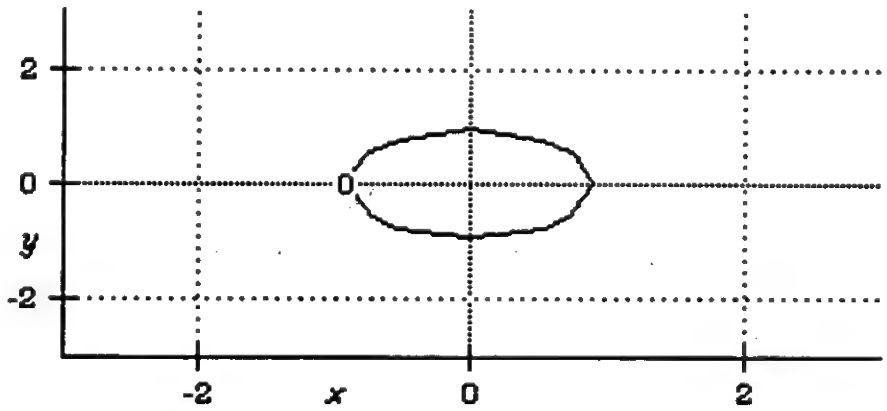
Select the zero on the left hand side, and type "`zero`". Upon the first keystroke, Theorist makes a new assumption below the conclusion.

The result is:

$$\text{zero} = -x^2 - y^2 + 1$$

With the blinking cursor still in this equation, pull down the **Graph** menu and from its **z = f(x, y)** submenu, choose **Zero Contour**.

When asked about "zero," declare it as a variable. Theorist displays another dialog box, this one inquiring about variables; simply press Return to accept the defaults. Your graph then appears:



This should be a perfect circle, but it is an oval because the default graph stretches to fit within the viewport. You can fix that by doing one or both of two things:

1. Resize the graph viewport, using the size box in the lower right corner, so the viewport is a square. (This may take a few tries.) When the grid rectangles are square, the proportions are right.

2. Open the details and change the aspect ratio pop-up menu from **"Stretch To Fit"** to **"True Proportions"**. This guarantees the proportions are always equal; in other words, one unit vertically is one unit horizontally.

This graph is a special kind of contour plot, where the only contour drawn is for the zero value. Theorist evaluates the variable "zero" all over the graph surface and draws the contour line or lines showing the boundary where it changes sign, no matter how many lines or how often it changes sign.

If you want to see a smoother graph, click on the More Accuracy button (with the finer mesh). For a faster (but coarser) graph, click on the Less Accuracy button (with the looser mesh).

For more details on this kind of graph, see the Theorist Reference Manual.

COORDINATE MAPPING

In this tutorial, you will see how to take a graph in one coordinate system and map it to a completely different coordinate system.

When you map a graph to a new coordinate system, you end up with a graph that bears a resemblance to the original graph, but is distorted. That's the whole idea behind the new coordinate system.

A linear coordinate system can take the original image and move it, rotate it, and do certain kinds of uniform stretching on it. You can do a linear transformation by simply multiplying and adding matrices.

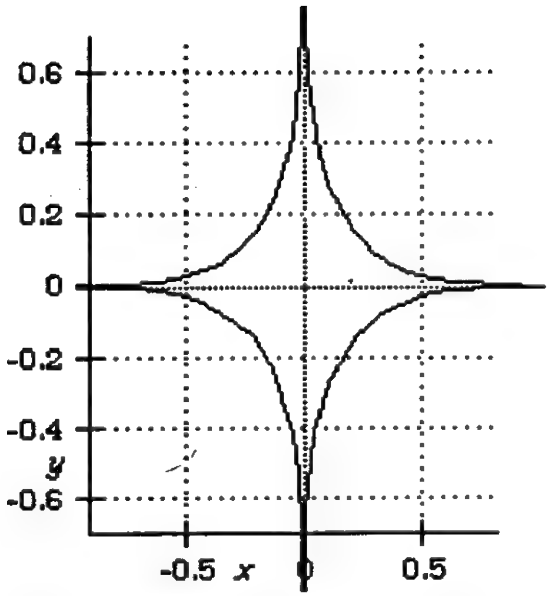
With this tutorial, you use a nonlinear coordinate system known as Polar Coordinates. The resulting distortion is interesting.

First, let's make a 2-D graph of a 4-point star. Enter the equations:

$$x = [\cos(t)]^5$$
$$y = [\sin(t)]^5$$

Declare *t* as a variable. Select both equations and choose the first item, **x = f(t), y = g(t)**, from the **Graph** menu to make a parametric graph. When Theorist puts up a dialog asking about dependent and independent variables, make sure *x* is the first dependent variable and *y* is the second one.

When the graph appears, adjust the view with the hand if necessary.



Graph Icon

Let's make another graph with this same star mapped through polar coordinates. Click on the Graph icon. Perform a Copy then Paste. This duplicates the graph and places the copy below the original.

Now, from the bottom graph's details, find the line plot:

∿ Line at (x,y) where $t = -3 \ldots 3$ with a [normal] line.

Our next step is to adjust (x,y). Click down so the blinking cursor appears just before the first parenthesis in (x,y). Type "`FromPolar`", exactly with initial caps and no space between From and Polar:

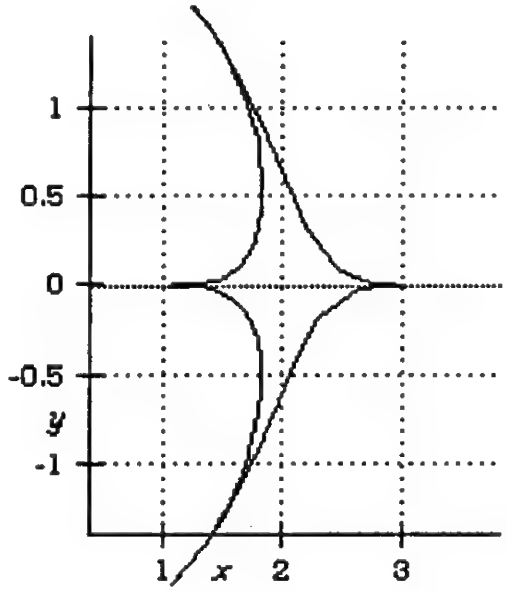
∿ Line at `FromPolar`(x,y) where $t = -3 \ldots 3$ with a [normal] line.

Choose **Clarify** from the **Notebook** menu. When the dialog appears, click **PreDefined** or press Return to declare FromPolar as a predefined function. Now the graph looks like a mess, but don't worry about it. It's a mess because the center of the star is right on top of the origin. The origin, in polar coordinates, is a very strange place where distortion hits an all-time maximum. Let's move the center away from the origin...

The star, which was in the (x,y) rectangular coordinates, is now in the (r,θ) coordinate system. (You still use the variables x and y but the way you are looking at it, the x is interpreted as r and the y as θ). You can improve the appearance of your graph by moving it 'up' in the r direction. That is, by adding something to r (the first coordinate).

Hand Cursor

Change FromPolar(*x*,*y*) to FromPolar(*x* + 2, *y*). You will have to adjust your view of the graph with the hand and/or the rocketship because the graph is off to the right. It looks like this:



Exercise

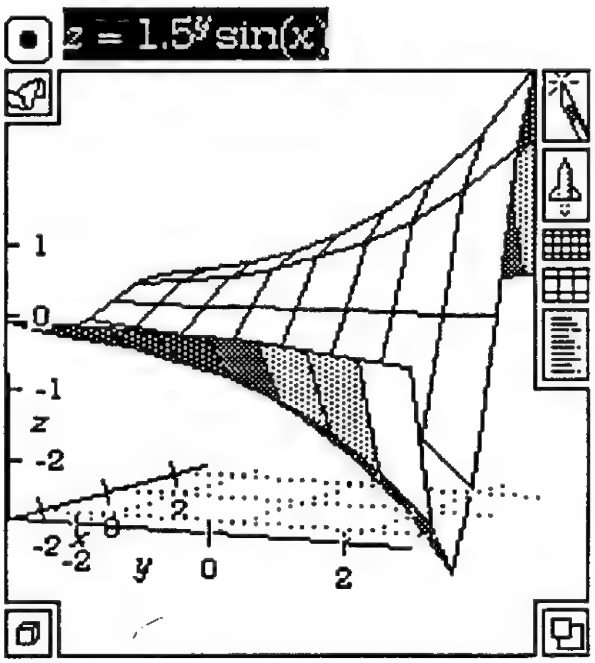
Change the 2 to 3 or 4 to move the star's center further away from the origin. Slide it around by adding something to the θ coordinate (actually *y*). This rotates it around the origin.

3-D GRAPHING

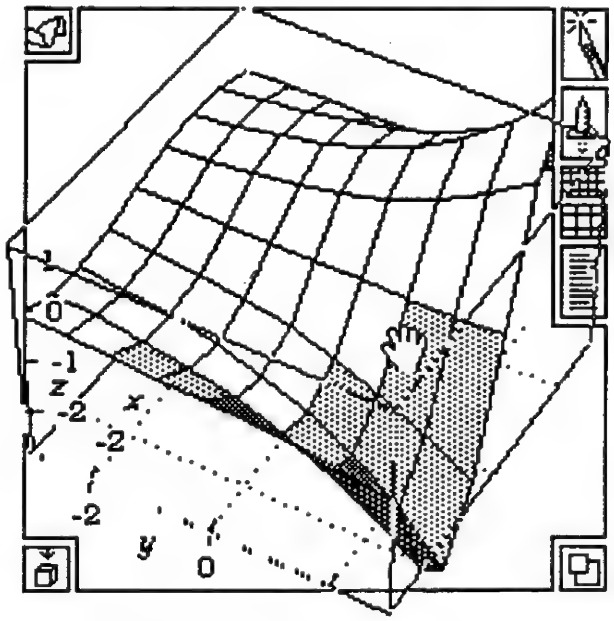
Type this formula into a new notebook:

```
z=1.5^y Esc *sin(x)
```

Click on the equal sign so that the equation is selected. Then pull down the **Graph** menu and from its **z = f(x, y)** submenu, choose **Color 3D** (yes, even if you have a black-and-white Macintosh). A dialog box appears asking you to confirm the independent and dependent variables you want for your graph. Confirm them and click **OK**. Then the graph appears:



As with two-dimensional graphs, the cursor becomes a hand for scrolling the graph. This time, though, what it does is a bit different. The graph as drawn is in a rectangular box, usually invisible. With the scrolling hand, you can rotate this box in space to get a better view:

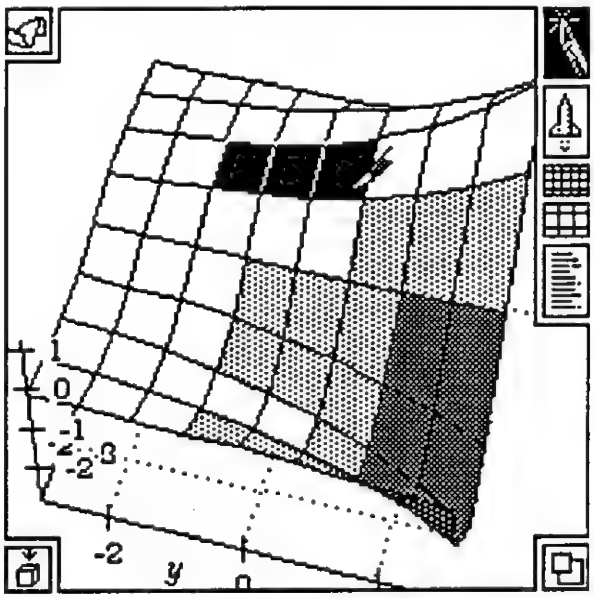


The hand rotates the cube in much the same way the palm of your hand rotates a globe of the world.

Orientation Guide (showing top view)

The knife also works differently in 3-D than its 2-D counterpart. To see how it works, rotate the graph so you are looking down at the top of the surface (or up from below). This process works best when you are looking at a surface perpendicularly (although you can play around with it later).

Click on the knife, as before. This time, you slice out a section of the graph by dragging on the surface itself. The piece you slice out constitutes an integral number of whole panels:



The tutorials "Inside Graphs," "Getting Familiar with the 3-D Space," "Coordinate Mapping," and "More than Three Dimensions" have more information on 3-D graphing. See the Theorist Reference Manual for further details.

GETTING FAMILIAR WITH THE 3-D SPACE

Surface Plot Icon

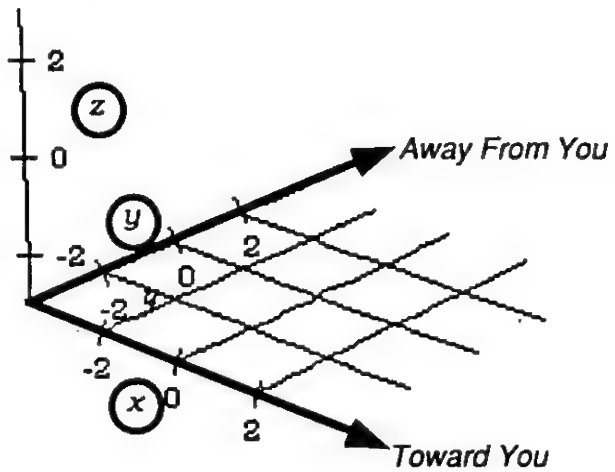
In a new notebook, make an Illuminated 3-D graph with the equation

$$z = \frac{x + y}{2}$$

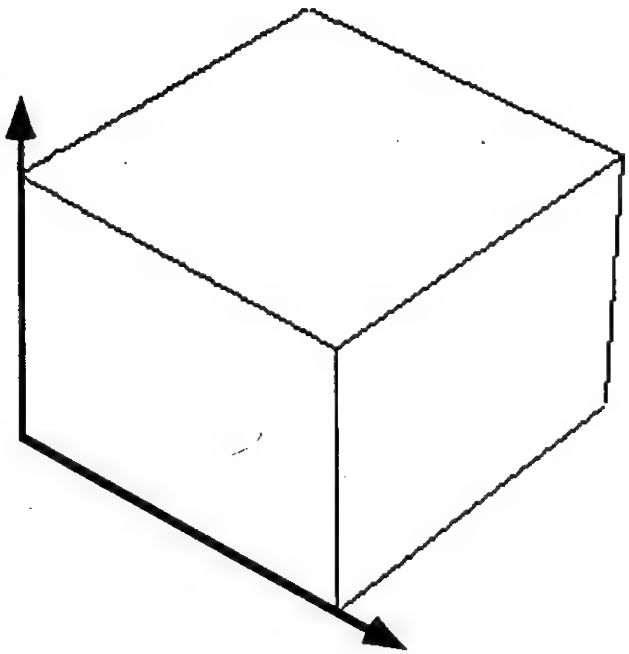
Open the graph details and delete the surface plot, then close the details. Delete the entire equation proposition because all we need is an empty 3-D graph.

Right Hand (Palm Up)

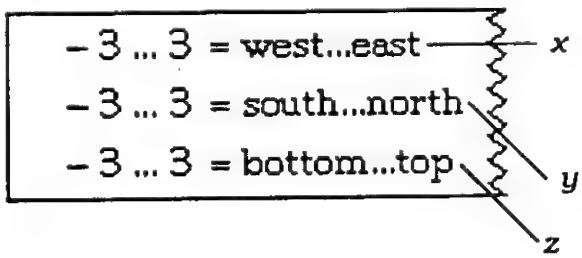
Note the coordinate axes. You have a right-handed coordinate system. If you take your right hand and label your thumb and forefingers *x*, *y*, and *z* in order, you get the correct orientation when you spread your three fingers apart like a tripod. Left-handed coordinate systems are mirror images of these.



Rotate the graph with the hand. Note that the axes all line up with the ghost cube.



The cube outlines a piece of three-dimensional space called the data region for that graph. The edges of the data region are the bounds. You can see and edit the bounds by opening up the details of the graph.



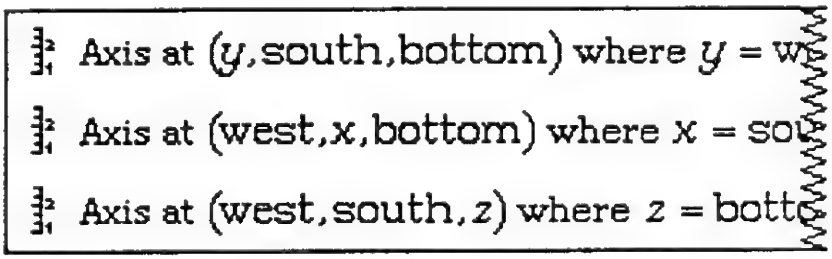
Since 3-D graphs are designed to work for all variables, not just *x*, *y*, and *z*, these bounds are referred to by geographic names. If you orient the axes on end, you notice that the west, south, et cetera all work as on a map or on the surface of the earth, with the top higher in altitude than the bottom. Note that the south, west, and bottom numbers are always lower and that the north, east and top numbers are always higher. Remember this. Think of Los Angeles (southwest) as being negative and New York (northeast) as being positive.

Expose and examine the Declarations comment within the graph details. Beneath it are six name declarations, one for each of the bounds numbers.

A Constant named west behaves as 3D X-Minimum .
A Constant named east behaves as 3D X-Maximum .
A Constant named south behaves as 3D Y-Minimum .
A Constant named north behaves as 3D Y-Maximum .
A Constant named bottom behaves as Vertical Minimum .
A Constant named top behaves as Vertical Maximum .

Within the graph these names behave as constants that always evaluate to the bound value of their name.

Also under the graph declarations are three axis plots, one for each dimension:



Each one consists of a vector that would be (west, south, bottom) except that one of the components has a variable in its place. The location (west, south, bottom) is the low point within the bounds because each coordinate is at its minimum there. Also, all the axes start here, but only because of their current set-up.

Click on the first axis icon, Copy, and do three Pastes. Change half of the souths and bottoms to their opposites so you have all four combinations:

x, south, bottom
x, south, top
x, north, bottom
x, north, top

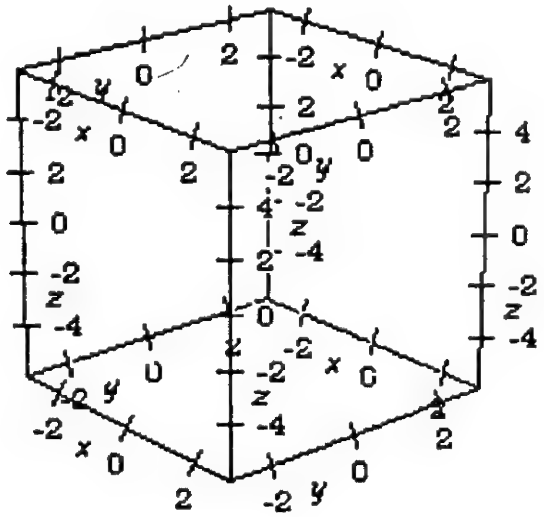
Your graph now has four *x*-axes. Do the same thing to the *y* axis:

west, *y*, bottom
west, *y*, top
east, *y*, bottom
east, *y*, top

...and the *z* axis:

west, south, *z*
west, north, *z*
east, south, *z*
east, north, *z*

Delete the grids, then take a look at the graph. You get a tricky, yet educational, result. You also get a fuzzy cube.



This cube of axes is good because you get a better idea of what is going on in the space.

All plots in your 3-D graph are drawn in the graph space. The viewer can see anything that is visible in the viewport, not only in the data region but beyond it. (For efficiency, Theorist does not draw objects that are more than a few diameters away from the data region.) You can rotate this space around and view any part of the space by typing its coordinates into the bounds.

Stretch to Fit
True Proportions

Aspect Ratio Pop-up Menu

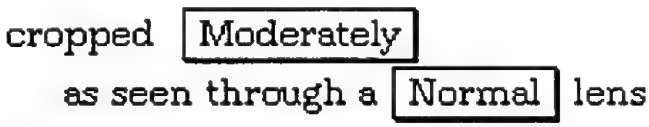
The aspect ratio controls how Theorist reacts when your bounds are not the same size. To see how this works, change your "bottom...top" bounds from "–3...3" to "–1...1". Examine the graph as you rotate it with it stretched to fit, then rotate and examine with true proportions.

"Stretch to Fit" means the space might be distorted along one or more axes in order to make the data region appear to be a perfect cube. Angles and slopes are distorted. This setting is best if your dimensions are not measured in the same units (for example, pressure versus temperature).

"True Proportions" means that one unit in the *x* direction corresponds to one unit in the *y* and *z* directions, such as when you are drawing geometric diagrams in the space where *x*, *y*, and *z* are all in units of length.

As you rotate a truly-proportioned graph, you may get a rectangle because the data region is not truly cubic. But, as you rotate a stretched-to-fit graph, there is always a distortion in effect that causes it to look like a perfect cube. You can see this by watching the tick mark spacing on the axes.

Looking at the graph through the viewport is like looking through the viewfinder of a camera. Theorist makes it easy to control all the options with two pop-up menus, one for cropping and one for viewing:



The cropping determines how close the image gets to the edge of the viewport. Since your data can be doing anything, you can set this on Very Wide, to see everything. (Do this now.) The image is shrunk so it all fits in. This option is particularly useful if your data has singularities or goes way outside the data region.

On the other hand, you may want to get a more close-up, dramatic view of some small part in the middle. You can get to a desired view in very tight steps if needed. Do this and experiment.

The lens setting determines the degree of perspective. The lowest setting is Very Wide Angle, which almost always distorts a lot. Choose this now. The main distortion is that close objects get very large. Therefore, widen the cropping at the same time.

Distortion lessens toward the other end. The last few are barely distinguishable on the screen but are visible on the LaserWriter. Try Telephoto now.

The last setting, Infinitely Distant, removes all perspective distortion. All parallel lines remain parallel. Try it now.

You can use the options described in this tutorial to customize your graph view to your exact needs.

MORE THAN THREE DIMENSIONS

Many areas of study explore systems with more than three dimensions. This tutorial shows a variety of ways of visualizing multidimensional mathematical data.

Enter the following equations into a new notebook:

$$y = \sin(x)$$
$$y' = \sin(2x)$$
$$y'' = \sin(3x)$$
$$y''' = \sin(4x)$$

When you get a declarations dialog, make the *y*'s user-defined variables.

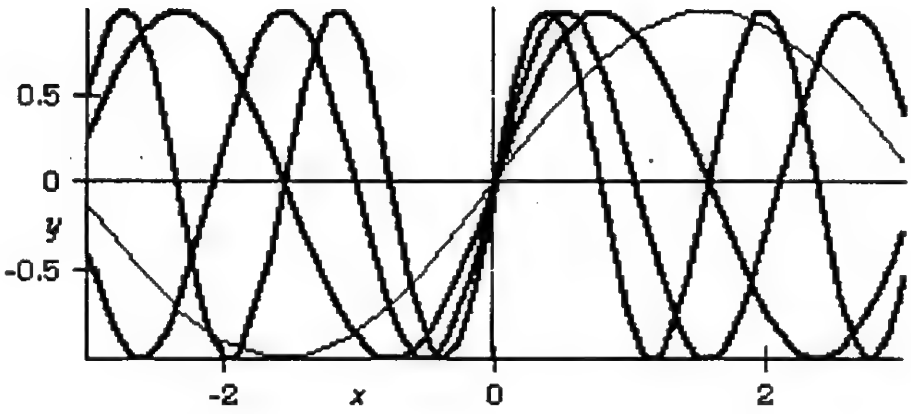
You want to look at all four of these at once, as a function of *x*. One way to do this is to make four graphs. Click anywhere in the first equation and press Command-G to create a graph, then repeat for each of the remaining three equations. Rearrange the graphs so they are stacked on top of each other like boxes.

This is one way of looking at all these functions, but it takes up lots of screen space, and you keep trying to visualize one graph overlayed on another. Making separate graphs as you now have is probably better for unrelated relationships, but in this case you want them all on the same graph.

When you get a dialog, just press Return.

Delete all four graphs. Click within the first equation and press Command-G to make a graph of it.

Click within the second equation and choose **Add Line Plot** from the **Graph** menu. A new, heavier line is drawn showing the second function. Similarly, click within the third equation, add another line plot for it, and do it again for the fourth function. (Unfortunately, you cannot use a multiple selection here.)



normal
heavy
dotted
dashed

Linestyles Pop-up Menu

The problem with this is the last three lines look the same. This may be all right. (Fortunately, in this case it does not look bad and it is obvious which function is which). You can change this, though, by opening the graph details and, with the pop-up menus, adjusting the linestyles for the four line plots.

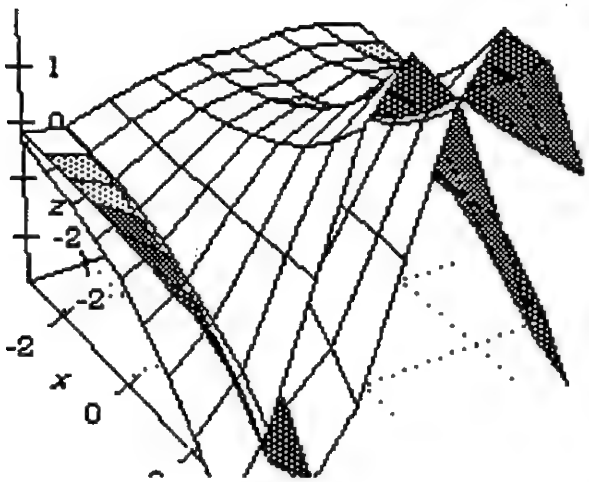
Something similar can be done with three dimensional graphs. Delete all your graphs and equations. Enter the following equations:

$$\square\ z = 1.5^x \sin(y)$$
$$\square\ z' = 1.5^x \cos(y)$$

When you get a dialog, just press Return.

As you might expect, click in the first equation and create an Illuminated 3-D graph, then click in the second and choose **Add Surface Plot** from the **Graph** menu. It helps visualization to rotate the whole graph for a better view. (On color-capable machines, it also helps if you make the second plot translucent.)

Note the intersections of the resulting graphs. (You may want to rotate it to get a better view.)

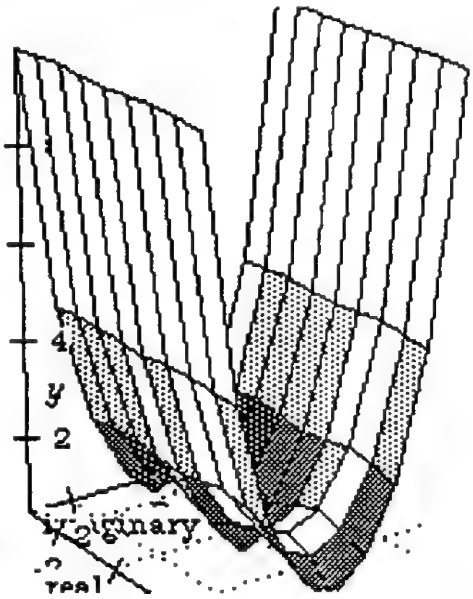


Delete everything. If your Mac is equipped with color, you can use color as an additional dimension. Even if you only have black and white on an SE, you can use Theorist's shading to tell you more about a relationship.

One way this comes up is in functions of a complex variable. Each complex dimension soaks up two of our mortal, real-valued dimensions. Therefore, a simple complex relationship $y = f(x)$ results in a four dimensional graph! Theorist has a way to grapple with this.

When Theorist asks you to declare *i*, make it predefined.

Press Return and type "`y=sin(x)`". This is the exponential function. It works all over the complex plane. You can witness firsthand what it looks like. With a selection anywhere in the equation, pull down the **Graph** menu and choose **Complex 3D** from the **y = f(x)** submenu. (You may want to resize and/or move the graph to get a better view.)



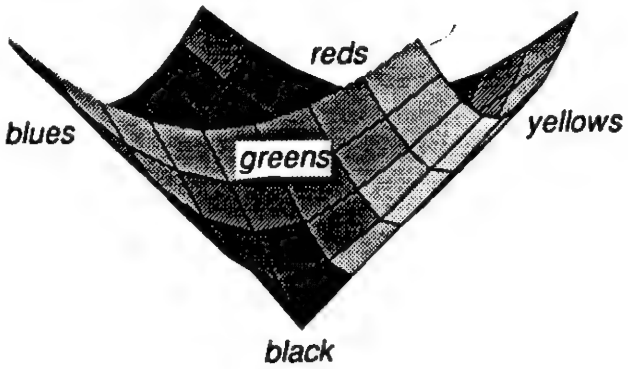
The complex 3-D graph uses the absolute value of the function for the height and the phase to indicate the hue. As you travel around the circle $|x| = 1$ in the complex plane, the color rotates around the rainbow. As the magnitude increases, the color bleaches to white, and as the magnitude is reduced, it fades to black. It's more spectacular in color than in black and white, because the color indicates the complex phase of the function.

If you are not using a color-equipped Macintosh, the next few pages may not be of interest to you. You may skip ahead to the Animation section a few pages ahead.

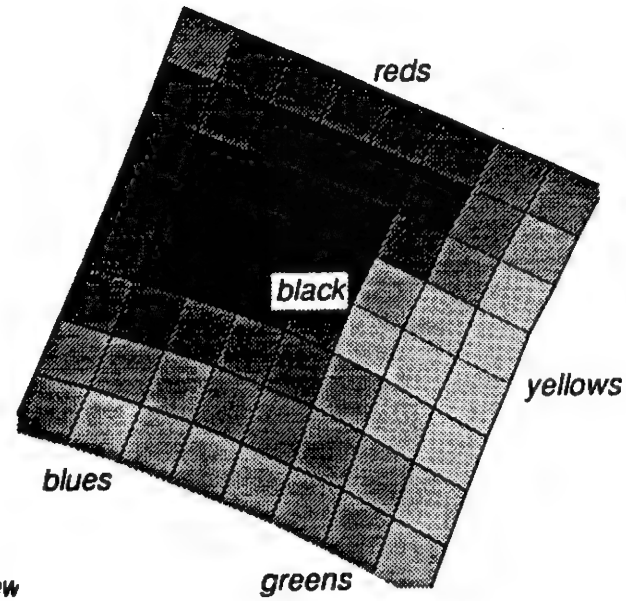
This can be seen graphically in the plot. The absolute value of $\sin(x)$ along the real axis proceeds as it does for real-valued $\sin(x)$, but along the imaginary direction, it turns hyperbolic:

$$\sin(ix) = i \sinh(x)$$

If all the above seems confusing to you, change the equation $y = \sin(x)$ to $y = x$. Then perhaps you can see how the complex 3-D graph does what it does.

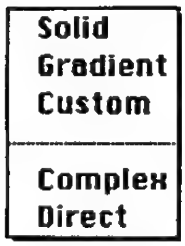


Side View



Top View

Open up the graph details and look at the surface plot. The surface plot is the most complex of the five kinds of plots because of all the coloring options.



Coloring Technique Pop-up Menu

The master switch for the color is a pop-up menu, set to Complex below.

There are five coloring options. Many of them use the value from the evaluated expression. In this case, Complex, the surface plot expects this expression to be complex-valued. If it was only real-valued, your colors would be more boring.

Change this pop-up menu to **Gradient**. When the graph is redrawn, surprise! Only the mesh is left. This is because Gradient expects the shade/color expression to be real-valued and it isn't. More specifically, it expects it to be in the range 0...1. Every pane for which the expression is not real Theorist makes transparent. To fix this, change the complex expression (which is under the Opaque pop-up menu) from y to $\frac{|y|}{5}$.

As the name implies, the color is a mixture of the two selected colors blended from 0 to 1. You can set an alternate minimum or maximum. Change the expression to $|y| = 1...3$. In this case, 1 becomes the minimum and 3 becomes the maximum.

Optical Properties Pop-up Menu

An even simpler color scheme is Solid. This option lets you choose which color is constant over the surface. It is most useful if you choose Illuminated from the first pop-up menu, which chooses optical properties of the surface.

Choose the **Custom** coloring option. This option is best when you need to discern levels of intensity. The graph is redrawn in white, blue, green, yellow, and black, according to the value of your expression, which, once again, ranges from 0 to 1 (or other limits you specify).

The complete mapping is:

Value	Color
0	Black
1/4	Blue
1/2	Green
3/4	Yellow
1	White

Colors in between are blended and/or interpolated. Colors beyond are pinned (black or white).

Notice the pop-up menu named Oregon. Custom color schemes all follow the above scheme ranging from 0 to 1, but they differ in the colors they use. The names are states in the United States. Most of them go from black to white through colors, but Kansas cycles through the rainbow. See the Theorist Reference Manual for a complete listing (or just try them out).

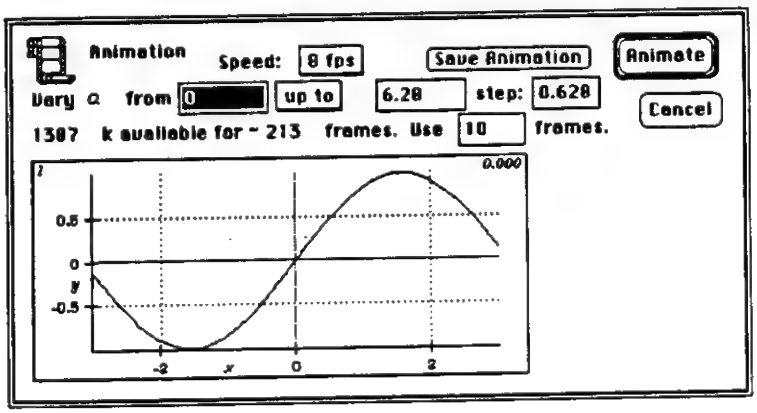
Choose **Direct**. This is a more serious option, where you want to directly calculate the color to use. This scheme requires you to enter a 3-vector into an expression. Change the $|y| = 1...3$ to Re(x), Im(x), 0.5:

(Re[x], Im[x], 0.5) determines the color components [Red, Green, Blue] .

Your surface is bathed in subtle shades. Maybe. You may have to use the knife to select such portions of the surface. See the Theorist Reference Manual for more details on this mapping scheme.

Animation

The final facility is probably the most fun. Delete your graph and all its equations. Type "`y=sin(x)`". Make a graph of it. Select the x and type "`+a`". Theorist can't find a value for a and does not draw the graph, but that's fine. Select a and choose **Animate...** from the **Graph** menu, which brings about this dialog:



Check the third line of the dialog, which is concerned with memory. Animation uses lots of memory, and this line tells you how many frames you have room for with the available memory. It lets you specify the number of frames to use in the animation. At least ten frames are required for a decent animation. If your dialog does not show enough memory available for at least ten frames, refer to the Animation tutorial in this manual or the Theorist Reference Manual to learn what you can do to remedy the situation.

Press Return. If all goes well, the graph redraws itself ten times. As it does, it saves the resulting images in memory. Watch the frame number in the upper left corner of the graph. Once ten frames have been created, Theorist starts animating the graph so that the sine wave moves along sideways.

You can stop the animation by clicking anywhere. You can change the parameters when the animation is idle.

Surprisingly, the complexity of the graph has little or no effect on the number of frames you have room for, although the more complex the graph, the more time it takes to save the images.

Unfortunately, to rotate the graph or change its colors or size, you have to cancel out of animation. For more information on animation, see the animation tutorial in this manual and the chapter on animation in the Theorist Reference Manual.

You can change anything about a graph with the animation parameter. You can rotate a graph by multiplying all your vectors by a rotation matrix. Let's do this now...

Cancel the animation and Delete the "+a" from your original equation. Enter the following new assumption:

$$\square A = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin[\theta] & \cos(\theta) \end{pmatrix}$$

Declare *A* as an M-linear op and θ as a variable. Open the graph details and locate the line plot and its vector. Select (*x*,*y*) and type "`*A`".

Now select any θ in the matrix and animate.

Exercise

Experiment with the speed and other animation options.

ANIMATION

Animation is a way of making a graph move. To animate, make a graph, modify its equations slightly, then choose **Animate...** from the **Graph** menu. This brings up the animation dialog, from which you can start, stop, and modify various aspects of your animation.

The process of animating even the most complex three dimensional, color graph is basically the same as the process of animating a Saturday morning cartoon. You start with an image of the graph, then render many similar images, each drawn slightly different from the previous image. Finally, you quickly flip through all the completed images, or frames, giving the impression of motion.

You first modify your graph by incorporating an extra parameter, the "time" or "animation" parameter, into some expression. (We usually use *a* as an animation parameter, but you may use any as-yet-unused variable.) This animation parameter changes in value from one frame of animation to the next.

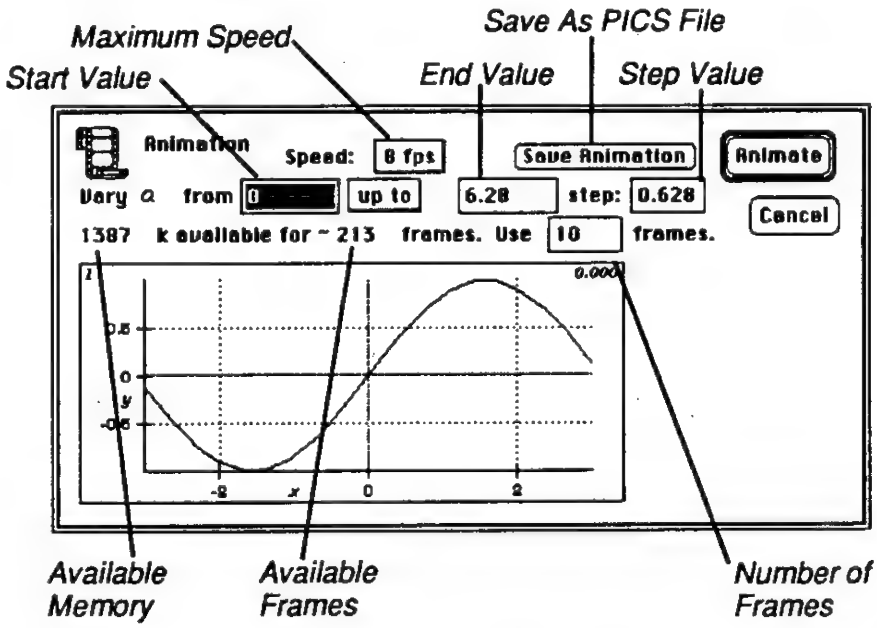
Start by animating a simple two-dimensional graph. Type:

```
y=sin(x)
```

Graph it by pulling down the **Graph** menu and from its **y = f(x)** submenu, choosing **Linear**. (Or just press Command-G.)

Next add the animation parameter. Select the *x* and type "`+a`". This will cause the phase of the sine wave to vary from one frame to the next when the animation begins. (Don't worry if, after typing +a, an error message appears in the graph area. This is taken care of later.)

The next step is to select the animation parameter, so drag over a. Pull down the **Graph** menu and choose **Animate...**.



Your next step is to check whether you have memory available for enough frames to make an animation. In the above dialog, Theorist says there are 1,387 kilobytes available for a maximum number of approximately 213 frames. The number 10 appears in the box for the number of frames to use; you can change this number if you want. Hopefully, your dialog says there is at least enough memory for ten frames. If you only have enough memory available for less than ten frames, change the number of frames to the suggested maximum number in your dialog. If your available memory allows over ten frames, you can change the number of frames, but it will take more time to animate, so leave it at ten for now.

Now that you have set the number of frames, click on the **Animate** button, or press Return. (Ignore the other options and controls in the dialog for now.)

The animation begins at a start value and increments by the step value for each frame until it reaches the ending value. The first time through the sequence, Theorist draws the graph slowly and carefully on the screen for each frame, saving frame images in memory as it goes along. After the last frame has been saved in this manner, the animation begins. To stop the animation, click on the **Animate** button again.

Memory Usage

If you are not using MultiFinder, Theorist tries to use all the extra available memory in your Macintosh for animation, and you have the potential for many frames. However, if you are using MultiFinder, Theorist can only work with the amount of memory allocated to it. So even if you have an eight-megabyte Mac, if you allocated Theorist a 600K partition, you don't have much to work with. You can change Theorist's memory partition by going to the Finder, selecting the Theorist icon, and choosing **Get Info** from the **Notebook** menu or pressing Command-I.

The number of frames you have space for also depends upon a couple of other factors. The larger the number of pixels in a graph viewport, the fewer the frames you will have room for. (You can resize the graph viewport outside the animation dialog to cut down on the number of pixels.) The larger the number of bits per pixel your Macintosh is displaying, the fewer the frames you will have room for. Macs running in black-and-white use only one bit per pixel. Macs running in gray scale or color modes may have eight or more bits per pixel and consequently have less room for frames than a black-and-white Mac. You can change the number of bits per pixel with the Control Panel desk accessory's Monitors device. Although color images are spectacular, they eat up memory quickly!

Virtual memory may or may not improve your situation. It works best when you have a huge program which does not all have to be in memory at once. The image data for animation *does* have to be in memory all at once. Therefore, your animation may be slowed down by your hard disk.

Image Calculation and Rendering Speed

For more complex graphs, you may notice that it takes Theorist some time to calculate and draw the graphs as it works at rendering and storing the images. The speed at which Theorist calculates and renders each frame depends mainly upon:

1. Complexity of the graph: How many plots? How complicated are the necessary calculations? What resolution setting is the graph at?
2. CPU speed and existence of math co-processor

The image size and the number of bits per pixel have negligible effects.

Image Animation Speed

With the **speed** pop-up menu in the animation dialog, you can set the speed at which Theorist flips through and displays the frame images. There are no calculations involved in this, the actual animation; the Mac spends most of its time simply throwing bit-map images up on the screen in rapid succession. The fastest setting is limited by three main factors:

1. Size of the graph, in pixels
2. Number of bits per pixel
3. CPU speed
4. Hard disk speed (if using virtual memory)

If virtual memory is in use and the hard disk is running constantly as it animates, you can increase the speed by reducing the amount of memory you are using for the image data (see above). Once it is below a certain threshold disk access drops dramatically and animation becomes much faster. It may take some experimentation to find this threshold.

The existence of a math co-processor is irrelevant, as is the complexity and resolution of the graphs. The images have already been calculated; all that is happening now is the machine is moving bytes around.

NUMERICALLY DEFINED SURFACE

In this tutorial, we define a function of two variables numerically and graph the results.

Typing hints:
`1,2,3;`
`A_?x,?y`

Enter the following equations:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$
$$f(x,y) = A(x,y)$$

Choose **Clarify** from the **Notebook** menu. Make A an M-Linear operator and f a function. Select both equations and choose **Make Working Stmt** from the **Notebook** menu.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$
$$f(x,y) = A(x,y)$$

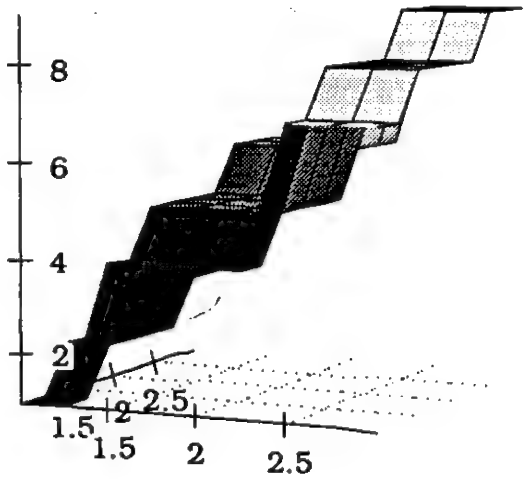
These two lines define a numerical function f of two variables. There are only nine values that f can return. The two arguments should be integers in the range 1 through 3. If they are not integers, they are rounded to integers.

The shadow letters are wildcard variables. It's as if some omnipotent being said "for all x" whenever you use x.

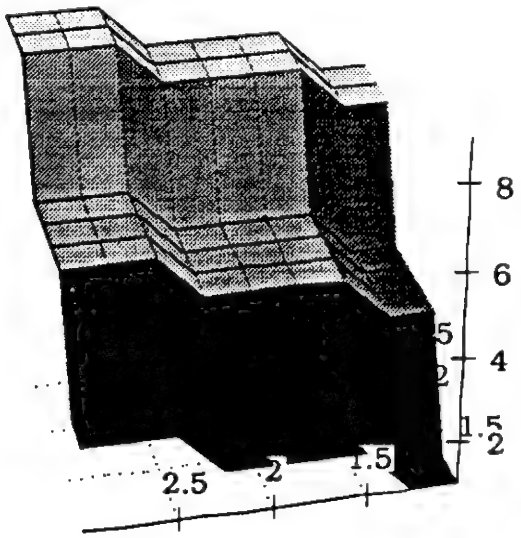
Press Return and enter the function:

$$z = f(x,y)$$

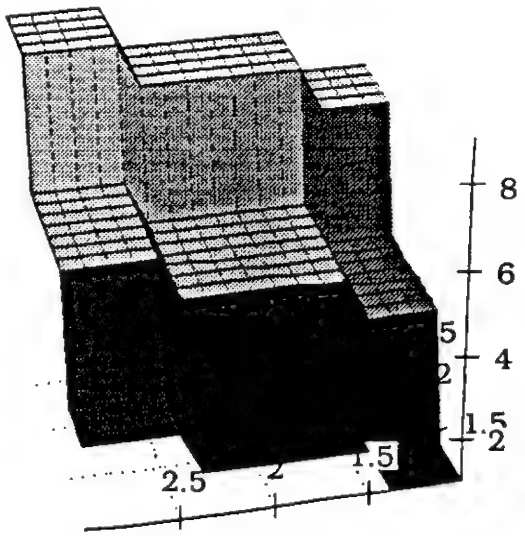
Then, with the selection still in the equation somewhere, create an Illuminated 3-D graph. It is not very informative:



Rotate it around so you are looking at the surface shown below:



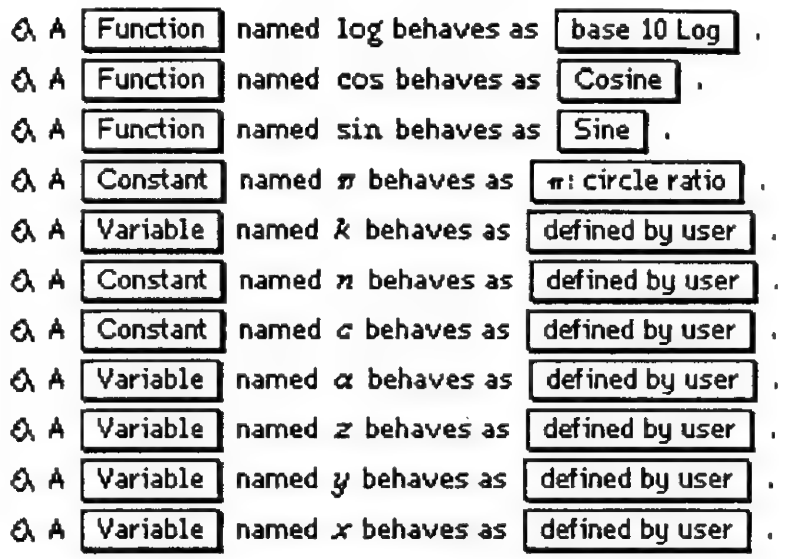
This function is discretely defined. To see this more graphically, click on the More Accuracy button, which gives you:



Names

NAME DECLARATIONS

Earlier in the manual, we mentioned there is something special about the top comment, Declarations. Double-click on the comment icon to expose what is beneath it:



Each of these declares a name you can use in your Theorist notebook.

Each part of a declaration specifies some facet of operation for that name. The pop-up menus are used to specify the class and the behavior.

Behavior

The behavior can be "defined by user" or it can be one of dozens of special behaviors. For instance, press Return and type in the name "`Fred`". Press Command-space to Clarify, and when the dialog comes up, specify user-defined. Afterwards, your name declaration for Fred appears under the Declarations comment:

Using the behavior pop-up menu, which currently says **defined by user**, change Fred's behavior to **I: square root of –1**, which is under the **Constants** submenu.

Now close the Declarations (double-click the icon again) and work on the "Fred" expression. Type "^2" after "Fred":

$$\square\ \text{Fred}^{2}$$

Select the expression and either Calculate or Simplify it.

$$\triangle\ \text{Fred}^{2} = -1$$

This is not just an alias, as with some computer languages. As far as Theorist is concerned, "Fred" *is* the name of the imaginary number.

Press Return and type:

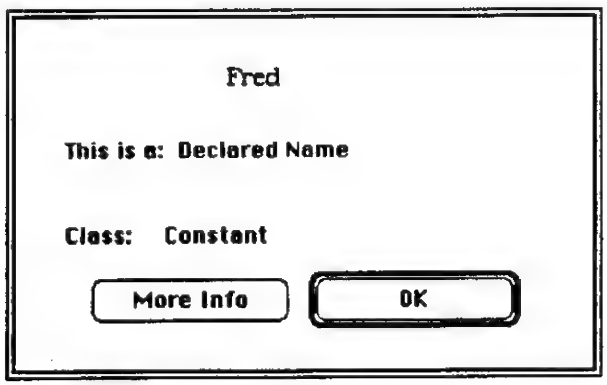
```
sqrt(-1)
```

Select the resulting proposition and Calculate it:

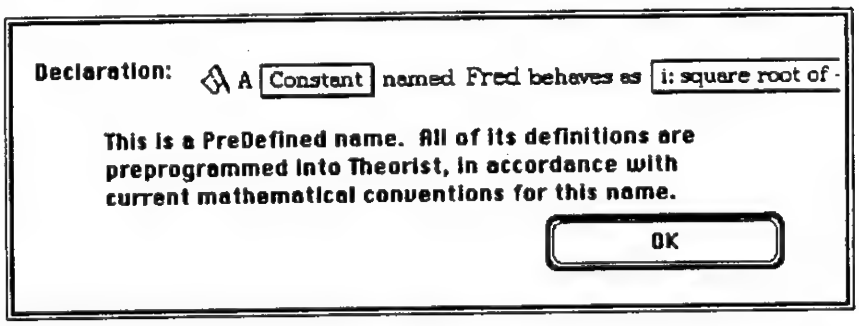
$$\triangle\ \sqrt{-1} = \text{Fred}$$

Now, if you want, you can have the name *i* refer to something completely different. You can use it to represent the current in a wire, for instance.

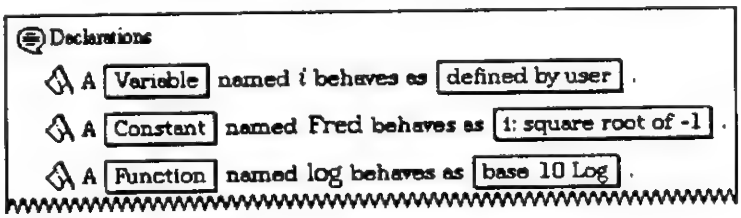
Select any Fred in the notebook and choose **Get Info** from the **Notebook** menu.



Click on **More Info** in the dialog.



Click on the declaration part of the dialog, and Theorist goes to the Declarations part of the notebook and selects the declaration for Fred.



When you change a name in a declaration all occurrences of that name change instantly.

Change the name Fred to Sally. Watch Fred change to Sally everywhere:

$$\triangle \sqrt{-1} = \text{Sally}$$

Legitimate names may consist of any upper and/or lower case letters and any Greek letters and other symbols from the Greek sub-palette. They may not contain digits, underbars, or other special characters. See the Theorist Reference Manual for more details.

Name recognition is case-sensitive. Names can be multiple characters; single-letter names are always displayed in italics. Single-character names are preferable.

Type the following, paying particular attention to upper and lower case:

```
a=Sin(x)+SIN(y)+sIn*(z)
```

Clarify the notebook. Note that Theorist asks you to declare Sin, SIN, and sIn, because it recognizes none. It does not confuse them with the predefined sine function, "sin" (all lower case).

Class

A name's class determines some of its properties.

Type "`d* (A*B)`" so you have the following:

$$\square\, d(AB)$$

Clarify and declare *d* predefined, *A* as a constant and *B* as a variable. Simplifying the expression gives you:

$$\triangle\, d(AB) = A\,dB$$

Using the method described previously (Get Info/More Info), change *A* to a variable. Simplify the original expression again.

$$\triangle\, d(AB) = B\,dA + A\,dB$$

Press Return and type "`(A+B) ^2`" so you get:

$$\square\, (A+B)^2$$

Expand it to get:

$$\triangle\, (A+B)^2 = A^2 + B^2 + 2AB$$

Change the name declarations for both *A* and *B* to M-Linear ops, which causes them to behave like matrices. Expand $(A + B)^2$ again.

$$\triangle\, (A+B)^2 = A^2 + BA + B^2 + AB$$

Because they do not commute, they do not combine.

Change *B*'s class to function. Notice how your expressions change. Functions tend to apply themselves to whatever they find nearby.

Press Return and type "`d(d[x*y])`" so you get:

$$\square\, d(d[xy])$$

The *d* operator takes derivatives. Its class is D-Linear op. Simplify the expression and notice the resulting expression has parentheses for grouping:

$$\triangle\, d(d[xy]) = y\,d\,dx + 2(dx)dy + x\,d\,dy$$

D-Linear ops are not necessarily associative. For instance, the expression *(dx)dy* is not the same as *dxdy*. In the latter case, the first *d* applies to everything to its right; in the former case, the first *d* is applied only to *x*.

Algebra

SOLVING LINEAR SYSTEMS

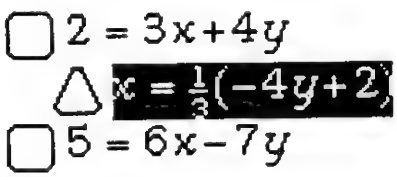
Whenever you have multiple equations with multiple unknowns, the general strategy is to eliminate variables one by one until you have just one equation in one unknown. Then you solve it. To eliminate a variable, take any equation, solve for the variable you want to eliminate in terms of the others, then substitute it into all the rest of the equations.

Enter these two equations:

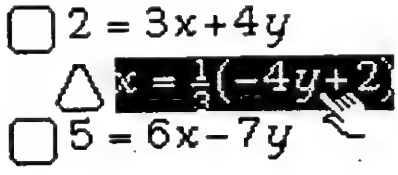
$$2 = 3x+4y$$
$$5 = 6x-7y$$

This is the traditional "two linear equations in two unknowns" problem. This tutorial shows you how to solve and manipulate equations with Theorist by walking you through this set of equations.

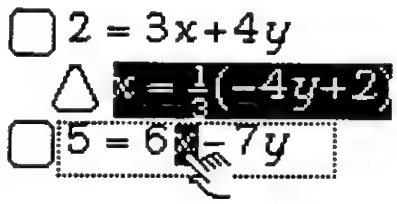
First, select the *x* in the first equation, and drag it as described in the Manipulating Equations tutorial to isolate it.



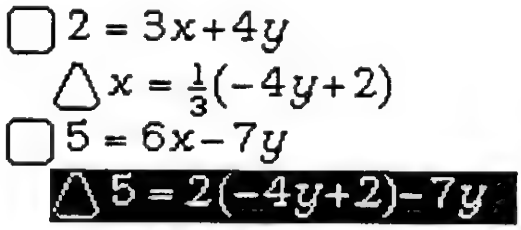
The next step of the derivation is to take the first equation (now rearranged as a conclusion) and plug it into the second equation. Make sure the equation for *x* is selected; if it isn't, click on its equal sign to select it. Hold down the Command key and notice that the arrow cursor becomes a pointing hand:



If you drag with this pointing hand, you end up dragging an outline of the equation around. If, while dragging this around, you happen to drag it over an occurrence of x, that x highlights, to indicate you can drop it in to perform a substitution. Drag it over the x in the bottom proposition:

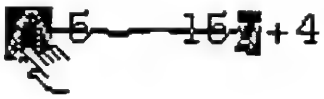


When you end the drag by releasing the mouse button, the substitution is performed:



With the last conclusion selected, choose **Expand** from the **Manipulate** menu to combine the two y's:

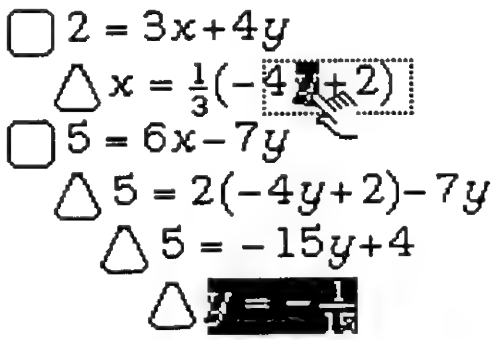
Now that you have one occurrence of y, you can solve for it by using Isolate, but this time do it by hand. First select the y and then hold down the Command key as you drag the y to the left, until it is over the conclusion icon, and release the mouse button:



This yields your answer for y:

Substitute your answer into the second equation from the top to get an equation for x.

Your answer is already selected, so hold down Command, and drag it to the location shown below.



This gives you:

Exercise

Substitute these answers for x and y into the original equation. Prove that it works.

Exercise

Use multiple selections to make your work easier. Redo the last exercise by first selecting both the equations for x and for y, and by dragging them over the proposition icon (*not* the variables themselves) for the target equation.

Enter the following two equations and solve for x and y.

$$a = 3x + by$$
$$c = px + 9y$$

The methods described previously are fine if you only have a few equations, but can get tedious if you have many linear equations. Consider this system of equations:

$$5x + 6y - z + w = 3$$
$$2x + 2y + 4z - w = 7$$
$$-4x + 3y + 9z + 12w = -4$$
$$y + 3z - w = 0$$

Solving this would involve throwing around several equations and plugging everything into everything and would be really tedious. The easy way to do it is to make it a matrix equation.

In a new notebook, type the following:

```
(5,6,-1,1;2,2,4,-1;-4,3,9,12;0,1,3,-1) (x;y;z;w)=(3;7;-4;0)
```

Clarify w as a variable, which leaves you with:

$$\square \begin{pmatrix} 5 & 6 & -1 & 1 \\ 2 & 2 & 4 & -1 \\ -4 & 3 & 9 & 12 \\ 0 & 1 & 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ -4 \\ 0 \end{pmatrix}$$

Now, simply isolate the vector with the variables:

$$\begin{pmatrix} 5 & 6 & -1 & 1 \\ 2 & 2 & 4 & -1 \\ -4 & 3 & 9 & 12 \\ 0 & 1 & 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ -4 \\ 0 \end{pmatrix}$$

...so you get:

$$\triangle \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 5 & 6 & -1 & 1 \\ 2 & 2 & 4 & -1 \\ -4 & 3 & 9 & 12 \\ 0 & 1 & 3 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 7 \\ -4 \\ 0 \end{pmatrix}$$

Then, either Expand twice, for slow algebraic answers, or Calculate, for faster numerical answers.

$$\triangle \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} \frac{35}{8} \\ -\frac{123}{40} \\ \frac{53}{40} \\ \frac{9}{10} \end{pmatrix}$$

$$\triangle \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 4.375 \\ -3.075 \\ 1.325 \\ 0.9 \end{pmatrix}$$

MAXIMUM AND MINIMUM

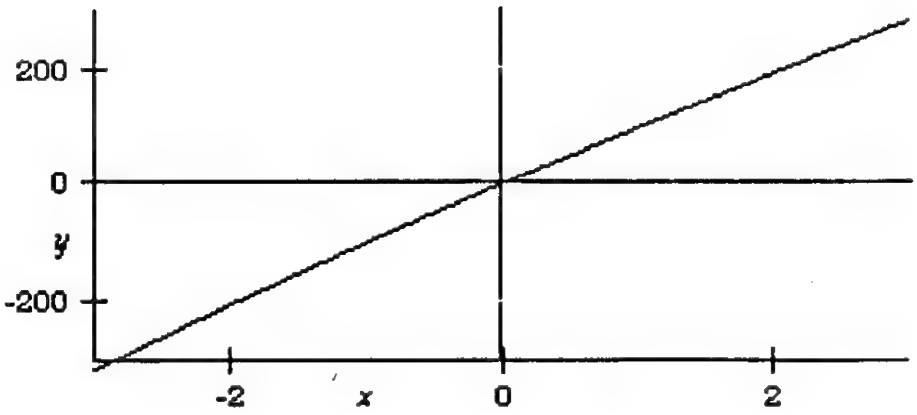
This example demonstrates how to differentiate an expression and solve an equation. Along the way you also use Substitute and Isolate, which you learned how to do in the previous tutorial, solving Linear Systems.

Type "`y=x*(100-2*x)`" to make a function of x

$$\square\, y = x(100-2x)$$

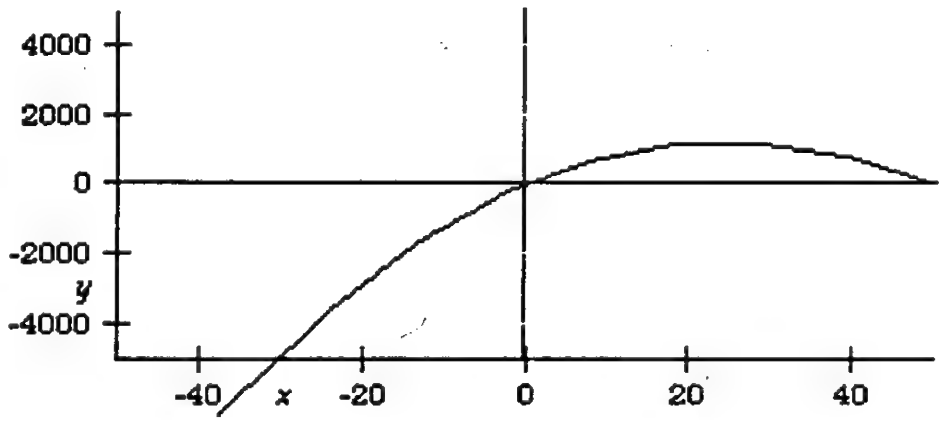
You want to know at what value of x your function obtains its maximum or minimum and what is the maximum or minimum value of your function.

To see what is going on, click anywhere in the equation and press Command-G. This graphs it:



Rocketship Button

This is not very interesting. Click on the rocketship button several times in order to zoom out and see what is going on.



You want to find the x and y values at the peak. This is the place where the derivative with respect to x of y is zero. It appears that the peak is near $x = 20$ and $y = 1000$.

As a shortcut, you can do the next three steps with: Option-6,**x**, Space,**y**.

Create a new equation by pressing Return. Click $\partial_y x$ on the function palette. This gives you:

$$\frac{\partial}{\partial ?}?$$

The highlighted question mark is your current insertion point. Type "**x**" and press Tab:

$$\frac{\partial}{\partial x}?$$

Notice how the selection changes as you press the Tab key.
Type "**y=0**":

$$\frac{\partial}{\partial x}y = 0$$

Click on the equal sign in your first equation. This highlights your equation.

$$y = x(100 - 2x)$$

Now hold down the the Command key and drag the highlighted equation over the y in the $\frac{\partial}{\partial x} y = 0$ expression:

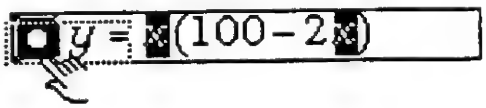
Theorist performs the substitution and differentiates your function:

$$\triangle \; -4x + 100 = 0$$

For the next step you want to solve this equation. Select x and choose **Isolate** from the **Manipulate** menu. This results in:

$$\triangle \; x = 25$$

For $x = 25$, the function $y = x(100 - 2x)$ reaches its maximum (minimum). To find what it is, substitute $x = 25$ into $y = x(100 - 2x)$. You do this in much the same way as in the previous substitution, except you drag over the assumption icon (the square) instead of the x (because you want to substitute for both x's):



Which results in:

$$\triangle \; y = 1250$$

QUADRATIC EQUATION

In this example we symbolically derive the well-known formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

...which is the solution of the quadratic equation $ax^2 + bx + c = 0$. Given a, b, and c, we will solve for x.

Make sure your preferences are set so that Auto Simplify and Auto Casing are both on, then enter the equation $ax^2 + bx + c = 0$, using your favorite method of equation entry.

Once you have the quadratic equation in your notebook, declare your variables by choosing **Clarify** from the **Notebook** menu. When Theorist displays a dialog for b, click on the **User Defined** button or press Return.

First select $ax^2 + bx + c$ by double-clicking an addition sign. Choose **Factor** from the **Manipulate** menu, which gives the answer:

$$\Delta\left(-\frac{1}{2}\sqrt{-4\frac{c}{a}+\frac{b^2}{a^2}}+\frac{1}{2}\frac{b}{a}+x\right)\left(\frac{1}{2}\sqrt{-4\frac{c}{a}+\frac{b^2}{a^2}}+\frac{1}{2}\frac{b}{a}+x\right)a = 0$$

But that would be too easy. Delete the answer and we'll derive it the hard way...

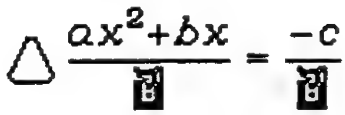
Select the c and choose **Move Over** from the **Manipulate** menu.

$$\Delta\ ax^2+bx = -c$$

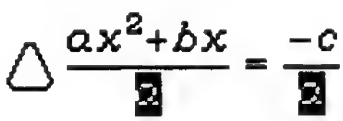
Next, we will divide both sides of the equation by a. Make sure the equation is selected as above and choose **Apply** from the **Manipulate** menu.

$$\Delta\ ax^2+bx = -c$$

Click on x/y from the function palette.



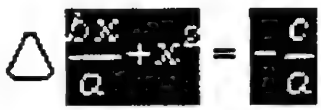
Click on a from the variables palette or type "a" from the keyboard.



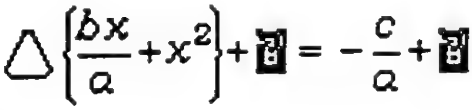
Click on the equal sign in the equation and choose **Expand** from the **Manipulate** menu.

We want to make the expression on the left of the equal sign into a perfect square. The missing term is the square of half of $\frac{b}{a}$, which is $\left(\frac{b}{2a}\right)^2$.

Let's add this term to both sides of the equation. With the equation selected, choose **Apply** from the **Manipulate** menu.

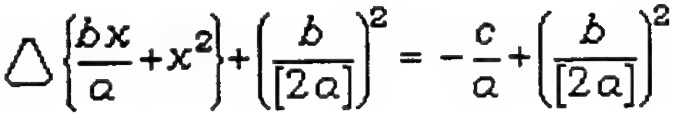


Click on $x + y$ from the Function palette, or type "+".



Typing hint:
`(b/(2*a))^2`

Enter the term $\left(\frac{b}{2a}\right)^2$ and you get:



Next, select everything on the left side of the equal sign and perform an Expand:

$$\triangle \frac{1}{4}\frac{b^2}{a^2}+\frac{bx}{a}+x^2 = -\frac{c}{a}+\left(\frac{b}{2a}\right)^2$$

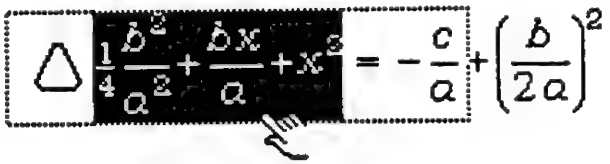
Now, create a new assumption. Enter the following, using your favorite method of equation entry:

$$\square \left(x+\frac{b}{2a}\right)^2$$

Select it and Expand:

$$\triangle \left(x+\frac{b}{2a}\right)^2 = \frac{1}{4}\frac{b^2}{a^2}+\frac{bx}{a}+x^2$$

Click on the equal sign to select the equation. Hold down the Command key as you drag it over the left-hand side of the previous conclusion:



...then release the mouse button to substitute it:

$$\triangle \left(\frac{1}{2}\frac{b}{a}+x\right)^2 = -\frac{c}{a}+\left(\frac{b}{2a}\right)^2$$

At this point, make sure the **Auto Casing** option is checked in the **Prefs** menu. Select *x* and choose **Isolate** from the **Manipulate** menu.

As you can see, Theorist saw there were both positive and negative square roots, so it put them in separate theories. (Theorist only does this when its Auto Casing option is checked in the Prefs menu.)

Although the answers are in acceptable forms, we want to put them into the form we are familiar with. Let's start working with the left proposition. Select everything inside the radical sign:

$$\triangle x = \sqrt{-\frac{c}{a}+\left[\frac{1}{2}\frac{b}{a}\right]^2}-\frac{1}{2}\frac{b}{a}$$

With the current selection, perform an Expand:

$$\triangle x = \sqrt{-\frac{c}{a}+\frac{1}{4}\frac{b^2}{a^2}}-\frac{1}{2}\frac{b}{a}$$

With the current selection, choose **Collect** from the **Manipulate** menu:

$$\triangle x = \sqrt{\frac{1}{4}\frac{-4ca+b^2}{a^2}}-\frac{1}{2}\frac{b}{a}$$

Select the entire expression to the right of the equal sign and do an Expand:

$$\triangle x = \frac{1}{2}\frac{\sqrt{-4ca+b^2}}{a}-\frac{1}{2}\frac{b}{a}$$

With the current selection, perform a Collect:

$$\triangle x = \frac{1}{2}\frac{\sqrt{-4ca+b^2}-b}{a}$$

Select the contents of the radical sign and choose **Commute** from the **Manipulate** menu. Select the whole numerator and choose **Commute** again, which gives the answer:

$$\triangle x = \frac{1}{2}\frac{-b+\sqrt{b^2-4ca}}{a}$$

INTRODUCTION TO POLYNOMIALS

What is a Nomial?

binomial:	$ax + b$
quadratic polynomial:	$ax^2 + bx + c$
cubic polynomial:	$ax^3 + bx^2 + cx + d$
quartic polynomial:	$ax^4 + ... + e$
polynomial:	$ax^n + ... + z$

A polynomial is an expression that takes on a special form. The form is a sum of powers of a variable (usually x), each being multiplied by some constant. Here are some examples of polynomials:

$x^2 + 2x + 1$ $\qquad$ $3x^3 + 2x^2 + x + 1$

$1 - x^{30}$ $\qquad$ $\frac{7}{3}x^2 + \frac{10}{13}x - \frac{42}{5}$

Polynomials are very useful because you can get a wide variety of different functions of x by controlling those coefficients.

The 'order' of the polynomial is the highest power present. In the four examples above, two are third-order, and two are second-order.

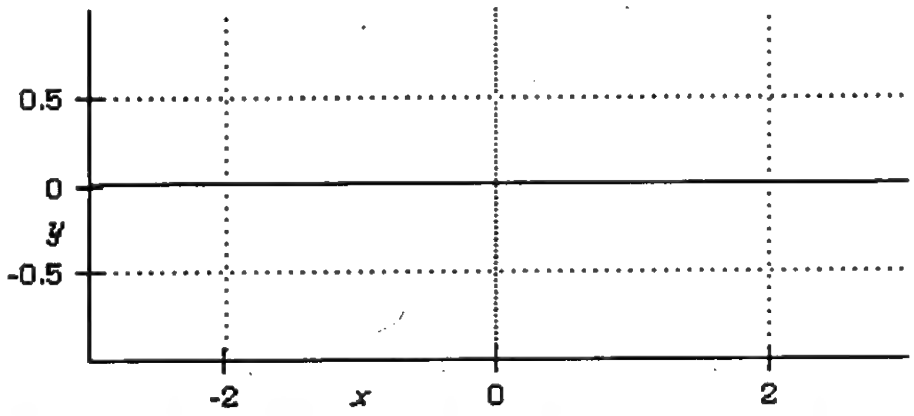
In a new notebook, type:

```
y=(0*x^3)+(0*x^2)+0*x+0
```

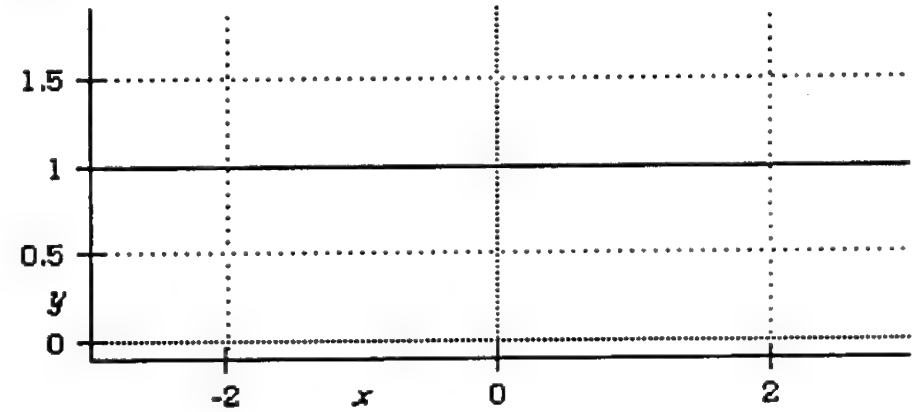
On the screen, Theorist gives you:

$$y = (0x^3) + (0x^2) + 0x + 0$$

Click anywhere in the equation and press Command-G to get a graph of it. A graph of zero looks pretty boring:

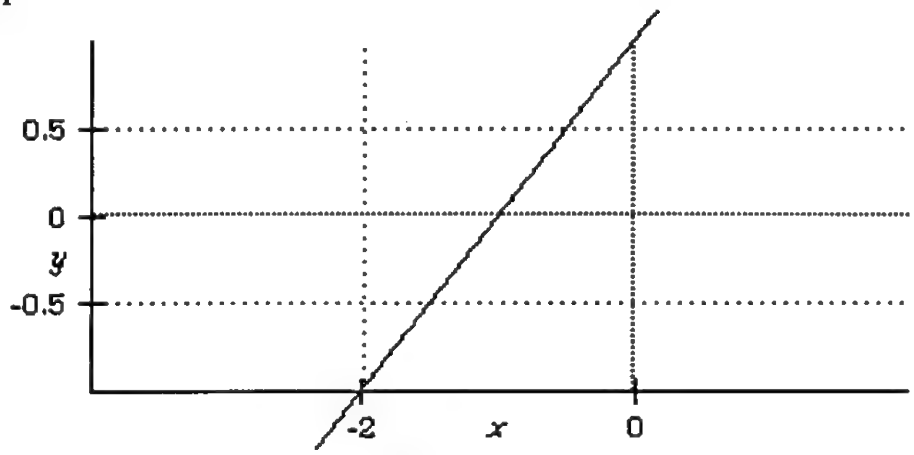


In the equation, select the last zero and type "1". The graph changes so you are looking at a zero-order polynomial. Use the hand to get a better view.



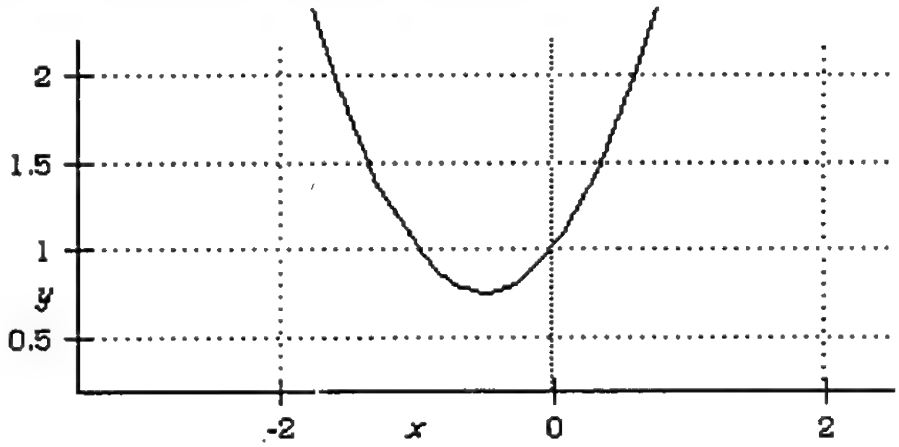
The zero-order polynomial is a constant value. Constant values are used for approximations for many things. They are not very accurate but they are extremely easy to use. By changing the coefficient to different numbers, you can change the height of the function everywhere at once.

Now change the next zero to the left, the $0x$ term, to one. You see, x really means x^1, just the same as $x^0 = 1$. Use the hand to center the graph.

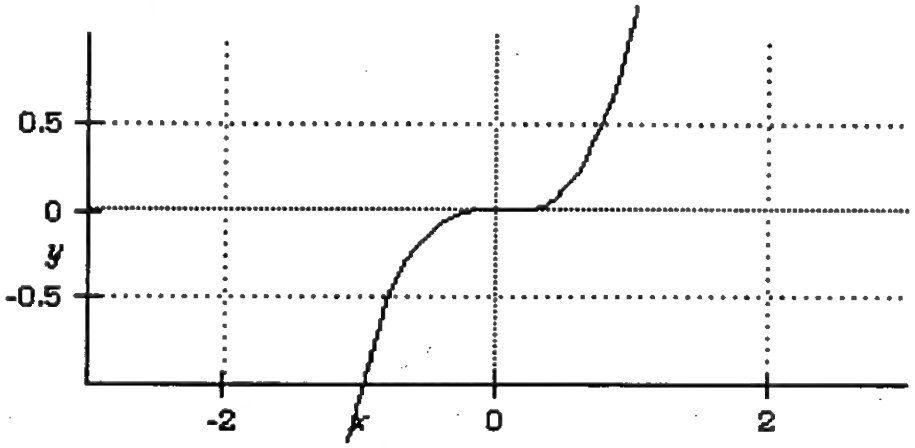


Now the function is a line. Straight lines are great for making approximations. You can change the slope (which is the tangent of the angle) of the graph by changing this first-order coefficient. As before, you can change the height by changing the zero-order coefficient. By varying these two you can get almost any straight line. The only kind you can't get is straight up and down. (Try it.) When you are done experimenting with these two coefficients, set them back to 1.

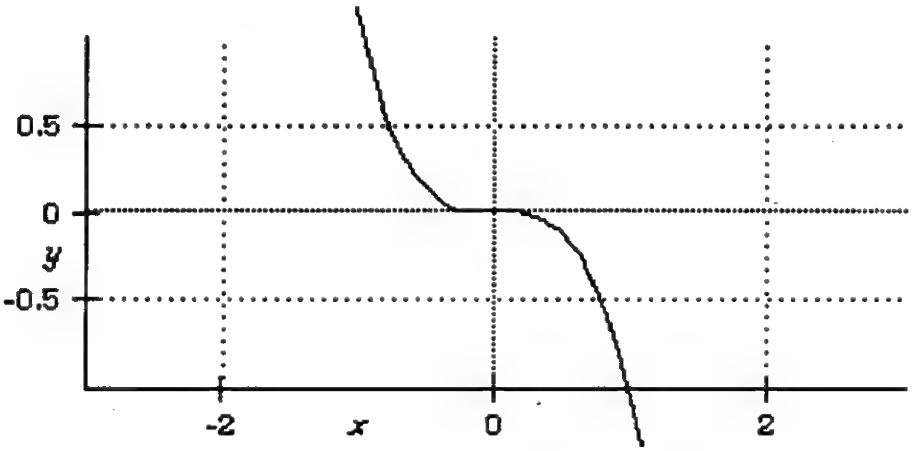
Now type "1" into the second-order coefficient, the x^2 one. Notice that this is the first time you get curvature. It always looks like a parabola, either up or down, depending on the sign of the second coefficient. (Use the hand again to center the graph.)

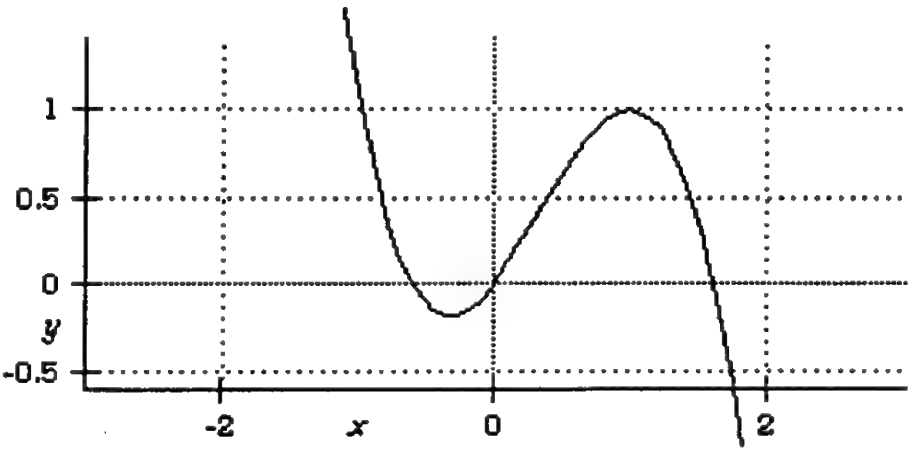


If you enter a 1 into the third-order coefficient, with zeros for the other coefficients, you get this:



If you then experiment with other combinations, you get graphs that look like the following:





Notice how the graph always has two humps, though they may be shallow. More terms and coefficients give the function more flexibility.

As you can see, the higher the order of the polynomial, the more degrees of freedom it has and the more different kinds of functions it can approximate. Later, in the Taylor Series tutorial, you learn how to generate a polynomial specifically designed to approximate a function.

MANIPULATING POLYNOMIALS

In this tutorial we go over many manipulations that are handy for use with polynomials.

Polynomials are found virtually everywhere in mathematics. They are so useful that some of the simpler symbolic algebra programs can represent nothing but polynomials.

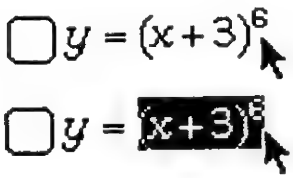
Polynomials form a world unto themselves. You can multiply two polynomials to get another polynomial. You can raise a polynomial to an integer power and get another polynomial.

Expand

Type:

```
y=(x+3)^6
```

Select the $(x+3)^6$ by clicking just below the 6:



Choose **Expand** from the **Manipulate** menu to expand the expression out.

$$y = x^6 + 18x^5 + 135x^4 + 540x^3 + 1215x^2 + 1458x + 729$$

Factor

Factor is the opposite of Expand. If it is not already selected, double-click on any "+" sign to select the expression after the equal sign. (Of course you can just drag over the whole expression.) Then choose **Factor** from the **Manipulate** menu (or press Command-8.)

Theorist takes a few seconds to do all its calculations; the rotating cube indicates Theorist is busy, like the busy wristwatch.

$$\triangle y = (x+3)^6$$

Factoring takes more time than Expanding because it is a difficult procedure akin to unscrambling an egg.

Collect

Collect is a stripped-down version of Factor. Press Return and type the following:

```
y=12*(x^2)-6*(x^6)+3*x^5
```

Select the right side of the equation. (Remember the shortcut: double-click on the "+" sign.) Choose **Collect** from the **Manipulate** menu (or press Command-7).

$$\triangle y = 3(-2x^4+x^3+4)x^2$$

Notice that Collect picks out $3x^2$ as a common factor to all the terms and it multiplies it on the outside. Also notice that it organized the inside in polynomial order with the highest power to the left.

Commute

Many people want their polynomials arranged in reverse order from the default. You can do this by selecting the polynomial $(-2x^4 + x^3 + 4)$ and choosing **Commute** from the **Manipulate** menu.

$$\triangle y = 3(4+x^3-2x^4)x^2$$

Two Variables

You can use Collect on even more complicated expressions. Press Return and type the following:

```
z=(x^3)*(y^2)+2*x*y+((3*x)/y)+((y^2)/x^2)
```

Press Command-space to remove the unnecessary parentheses. On the screen it appears as:

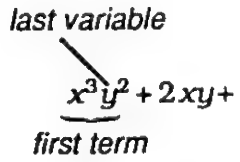
$$\square z = x^3y^2+2xy+\frac{3x}{y}+\frac{y^2}{x^2}$$

Select the whole expression to the right of the equal sign (drag or double click a "+") and perform a Collect command.

$$\triangle z = \frac{3x^3+(x^5+1)y^3+2x^3y^2}{x^2y}$$

Notice that it collects out a common denominator of x^2y. Anything in the denominator of any term is collected out as a common denominator.

Also note that it arranges the numerator as a polynomial in y, even collecting the x^5 and 1 together as a sort of coefficient for y^3. This is because y is chosen as the polynomial variable. You can control this. Theorist takes the last variable in the first term as the polynomial variable:

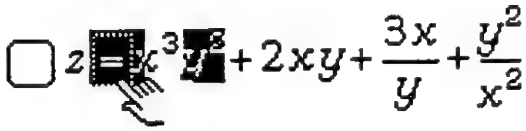


Let's have Theorist consider x to be the polynomial variable in the assumption. Start with this again:

$$\square z = x^3y^2+2xy+\frac{3x}{y}+\frac{y^2}{x^2}$$

This is a more direct way of Commuting.

Select y^2 as shown below and hold down Command as you drag the selection over to the left of the x^3. (The slot at the equal sign highlights.)



Drop in the y^2 by releasing the mouse button. This gives us:

$$\triangle z = y^2x^3+2xy+\frac{3x}{y}+\frac{y^2}{x^2}$$

Select the expression to the right of the equal sign by dragging or double-clicking a "+", then Collect it:

$$\triangle z = \frac{(2y^2+3)x^3+x^5y^3+y^3}{x^2y}$$

This time it collects terms assuming x is the polynomial variable.

More Than Two Variables

Almost any expression can qualify as a polynomial, but if you have too many variables, Theorist can get confused. Fortunately, you can do things more carefully by hand. Press Return and type `"(x+y)*(z+a)"`.

Select the whole expression and choose **Expand** from the **Manipulate** menu.

$$\triangle (x+y)(z+a) = ax+ay+xz+yz$$

If you try to factor this back with Factor, you get a confusing answer. Instead, do this three-step process:

❶ Select two terms that have a common factor, like this:

$$\triangle (x+y)(z+a) = ax+ay+xz+yz$$

Then do a Collect.

$$\triangle (x+y)(z+a) = ax+ay+(x+y)z$$

❷ Select the other two terms:

$$\triangle (x+y)(z+a) = ax+ay+(x+y)z$$

and do another Collect:

$$\triangle (x+y)(z+a) = (x+y)a+(x+y)z$$

❸ Select the expression to the right of the equal sign:

$$\triangle (x+y)(z+a) = (x+y)a+(x+y)z$$

and do another Collect:

$$\triangle (x+y)(z+a) = (x+y)(a+z)$$

Rational Fractions

A rational fraction is one polynomial divided by another. Create a new notebook and type "`((x^2)+5*x+6)/x+3`". Select the whole expression and perform an Expand.

$$\triangle \frac{x^2+5x+6}{x+3} = x+2$$

When you Expand a rational fraction, Theorist tries to divide the polynomials using long division. In this case, it comes out even. Change the 5 in the first proposition to 4. Select the expression and Expand it:

$$\triangle \frac{x^2+4x+6}{x+3} = 3\frac{1}{x+3}+x+1$$

This time it does not divide out evenly and there is a remainder.

Randomly chosen rational fractions usually do not divide out evenly; you will have a remainder. The order of the numerator of the remainder is always less than the order of the denominator. For instance, you never have a remainder that looks like this:

$$\frac{x^2+x+4}{x^2-x+3}$$

...because this can be divided one more step, which Theorist does for you.

You can reverse the long division and get back to the original rational fraction by simply selecting the whole expression to the right of the equal sign and doing a Collect:

$$\triangle \frac{x^2+4x+6}{x+3} = \frac{(x+3)x+x+6}{x+3}$$

One more Expand will give you what you started with. There are times when you need to break down a rational fraction even further. For Integration and Laplace transforms you usually need to have denominators that are binomials. Here is how to do it...

In a new notebook, type "((x^3)+(2*x^2)-3*x-7)/(x^2)-4" and press Command-space, which gives you on the screen:

$$\square \frac{x^3+2x^2-3x-7}{x^2-4}$$

The first step is to divide it out as before by selecting the entire expression and Expanding it:

$$\triangle \frac{x^3+2x^2-3x-7}{x^2-4} = \frac{x+1}{x^2-4}+x+2$$

Now select the denominator of the right fraction, $x^2 - 4$, and choose **Factor** from the **Manipulate** menu.

$$\triangle \frac{x^3+2x^2-3x-7}{x^2-4} = \frac{x+1}{(x+2)(x-2)}+x+2$$

Now select the whole right fraction and Expand. This time Theorist recognizes what you are trying to do and, instead of trying to divide the polynomials, does what is called a "partial fraction decomposition."

$$\triangle \frac{x^3+2x^2-3x-7}{x^2-4} = \left(\frac{1}{4}\frac{1}{x+2}+\frac{3}{4}\frac{1}{x-2}\right)+x+2$$

This is the simplest form into which Theorist can put your rational fraction.

ROOTS OF POLYNOMIALS

In this tutorial we discuss methods of finding roots of polynomials, or factoring them, which are equivalent tasks.

An expression is a group of symbols that evaluate to a number, matrix, or operator. An equation consists of two expressions connected by an equal sign. (Remember this distinction.)

One common task in mathematics is to find the value (or values) of x that makes a given equation true. These are the 'roots' of the equation.

Enter the equation:

$$x^2 + 1 = 2x$$

...using your favorite entry method.

Press Command-space to clean up any unnecessary parentheses. This gives you on your screen:

$$\square\, x^2 + 1 = 2x$$

(From now on in this tutorial, use your favorite method of equation entry and editing.)

Select the $2x$ and choose **Move Over** from the **Manipulate** menu. (Or, you can simply hold the Command key as you drag it to the other side of the equal sign.)

$$\triangle\, x^2 - 2x + 1 = 0$$

Remember: Expressions have zeros, not roots. Equations have roots, not zeros.

You can see that the roots of $x^2 + 1 = 2x$ are the same values that make the expression $x^2 - 2x + 1$ go to zero. These values are called the 'zeros' of the expression.

You can multiply polynomials together to get other polynomials. Make a new proposition and enter the expression:

$$\square\,(x-1)(x-1)$$

Select the whole expression and choose **Expand** from the **Manipulate** menu.

$$\triangle\,(x-1)(x-1) = x^2 - 2x + 1$$

Whenever you multiply two binomials, you get a quadratic. In fact, whenever you multiply an M-order polynomial by an N-order polynomial, you get an (M + N)-order polynomial.

Exercise

Find out what you end up with when you raise an M-order polynomial to the Nth power. Try Expanding $(x^3 + 1)^5$.

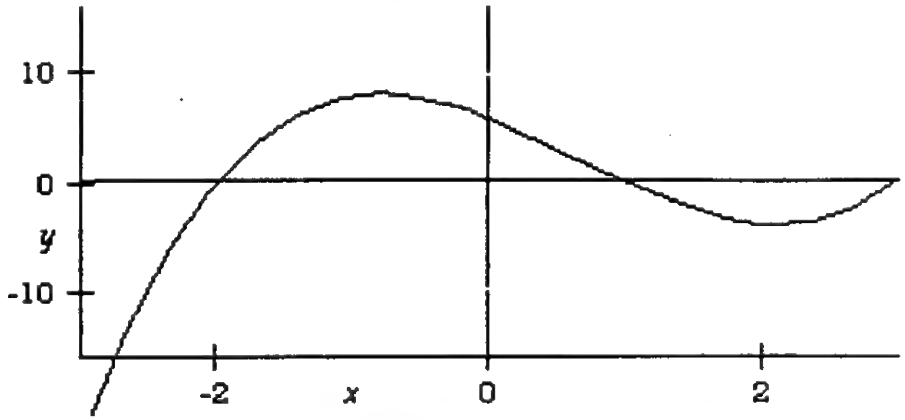
One way to think of polynomials is to factor them into binomials which, when multiplied together and expanded out, equal your polynomial. For example, $x^2 - 2x + 1$ could be thought of as $(x - 1)(x - 1)$ or $(x - 1)^2$.

An interesting side effect of this scheme is that the factors (the binomials from above) tell you of the zeros or roots of the polynomial. For example, create a new proposition and then enter the equation:

$$\square\,y = (x-1)(x+2)(x-3)$$

Take a look at the equation. If $x = 1$, it goes to zero. The same is true if $x = -2$ or if $x = 3$. Everywhere else it is *not* zero. Select the expression to the right of the equal sign and Expand it out.

Graph it (by pressing Command-G) to show where the roots are. (You may have to do some maneuvering with the knife, rocketship, and hand.) Notice the three zeros.



The fundamental theorem of algebra says that an N-order polynomial has N roots. Some roots might be complex. Some might be repeated. In fact, if all the coefficients are real, then the complex roots will come in complex conjugate pairs. For instance, the factors of $x^3 - x^2 + x - 1$ are $(x + i)(x - i)(x - 1)$.

The task of finding roots and factors of polynomials has been plaguing mankind since the time of the Greek mathematicians. In fact it is not always possible to find a root in the form that you want, especially for large polynomials, or those with multiple variables. We will explore a variety of methods of doing this. Fortunately, Theorist has many tools available to do this.

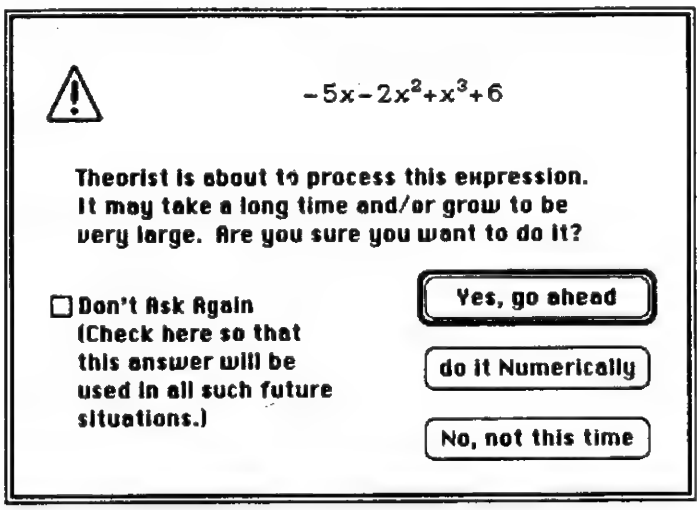
Factor Manipulation

In many cases, the fastest way to factor a polynomial is to use the Factor manipulation. In the equation:

$$y = x^3 - 2x^2 - 5x + 6$$

...select the expression to the right of the equal sign (remember that you can just double-click on the "+"). Choose **Factor** from the **Manipulate** menu.

At this point you will likely encounter the following dialog:



You can just press Return or Enter to continue. See the Theorist Reference Manual for details about this dialog.

(When you get a name declaration for *t*, make it predefined.) Theorist works at it and then gives your answer.

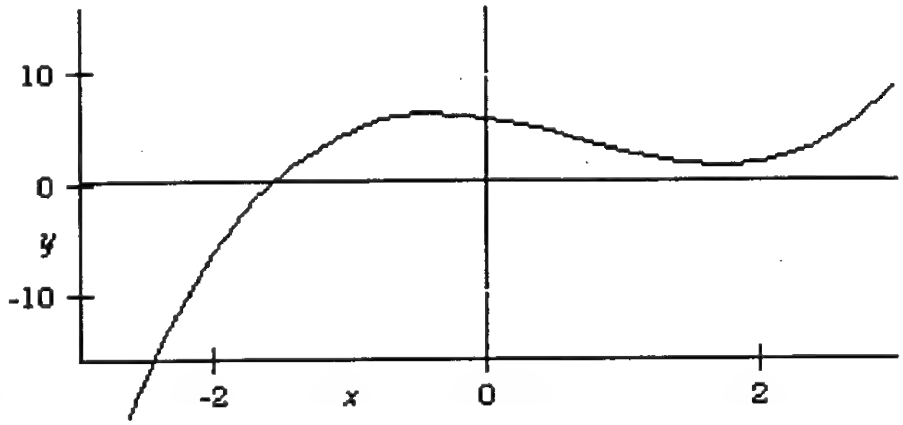
$$\triangle y = (x+2)(x-3)(x-1)$$

In some cases, your answer does not come out as cleanly. Change the 5 in the first conclusion to 2, which makes a new assumption:

$$\square y = x^3 - 2x^2 - 2x + 6$$

Choose **Make Working Stmt** from the **Notebook** menu, or press Command-D. The old working statement for *y* automatically turns off.

Play with the graph view so you get:



Notice that the graph no longer crosses zero in three places, just one. Changing the 5 to 2 raises the valley so it no longer dips below zero. The two roots look like they disappeared; in reality, they are on the complex plane. We will find them algebraically.

Now select the expression to the right of the equal sign:

$$\bullet\, y = x^3 - 2x^2 - 2x + 6$$

and Factor it. When the dialog appears, click on the Yes button. It takes a while and produces a large answer:

$$\triangle y = \left[x + \tfrac{1}{3}\cdot 100^{\frac{1}{3}} + \tfrac{1}{3}\cdot 10^{\frac{1}{3}} - \tfrac{2}{3}\right]\left[x + \tfrac{1}{6}\cdot 10\right.$$

By clicking Yes, we asked for the exact, analytical answer. If our polynomial had something other than numbers for coefficients, the result might be more useful.

Do this Factor over again, but when the dialog comes up this time click on the button for a numerical answer, which gives us an answer we can use:

$$\triangle y = (x + 1.5987)(x - 1.7993 + 0.71798i)(x - 1.$$

These roots seem unrelated to the original ones; two of them are complex. Most roots of randomly chosen polynomials are ugly like these. You can verify that these are, in fact, the correct roots. Simply do an Expand (you probably have the correct thing selected anyway).

Theorist can factor some rather large polynomials. Select the "6" in your polynomial and type "+x^6" so you get:

$$\square\, y = x^3 - 2x^2 - 2x + 6 + x^6$$

Then, try to factor the whole thing.

$$\triangle\, y = (x - 0.16923 + 1.492i)(x - 0.16923 - 1.$$

Note that in this case, Theorist does not ask you whether you want to factor it numerically; there is no choice. If your polynomial is cubic or quartic, Theorist uses the cubic formula or the quartic formula. Discovered during the Renaissance, these formulas always work, and the answer will always be exact, although it may be large and complex (like our answer with cube roots, above) and Theorist may take a long time to derive it.

If your polynomial has numerical coefficients, Theorist can find the roots using a numerical algorithm, even if the polynomial is larger than a quartic. The result, though, is not entirely exact, as we shall see. Expand this polynomial again to see if you get the original back, as you did before:

$$\triangle\, y = x^6 - 7.1557 \times 10^{-18} i x^3 + x^3 - 2$$

Note that it is not the same; there are several new terms. All of these are very small, in scientific notation. Many are imaginary. What happened was that the roots were computed to about 18 digits of precision, close to the maximum available in Theorist, which is 19 digits (the last few digits of a root are extremely hard to find). When you multiply them all back together, all of the coefficients that are supposed to be zero end up being zero, to within about 18 digits, meaning they end up being ugly tiny numbers like you see here. If you set your display precision to 19 digits, you will see that even the "good" numbers have some rough edges due to round-off error.

UnCalculate

Theorist has a manipulation to patch this up. Keeping your current selection, choose **UnCalculate** from the **Manipulate** menu. You will get your original polynomial:

$$\triangle\, y = x^6 + x^3 - 2x^2 - 2x + 6$$

UnCalculate examines all of the numbers in the expression and tries to figure out what they "should" be. Numbers that are close to integers, or to certain rational numbers, are made exact integers or rationals. UnCalculate also recognizes some square roots and other forms. (See the Theorist Reference Manual for more details on UnCalculate.)

Notice that UnCalculate will actually change the value of the expression slightly. You should be careful when using it. There are some situations where it is not appropriate to use it. Take for instance this expression expanded out:

$$\triangle(x-1\times10^{-5})^5 = x^5-5\times10^{-5}x^4+1\times10^{-9}x^3-1\times10^{-14}x^2+$$

If you UnCalculate it, Theorist looks at the very small coefficients and assumes they are supposed to be zero, and produces this result:

$$\triangle(x-1\times10^{-5})^5 = x^5-5\times10^{-5}x^4+1\times10^{-9}x^3$$

All of the extra terms are lost. If x is on the order of 10^{-5}, then this is not an appropriate thing to do, as these lost terms were significant and about half of the significant terms in the polynomial were lost. You must use good judgment when using the UnCalculate manipulation.

EIGENVALUES

In this tutorial you will learn some tricks for typing in matrices and you will learn some linear algebra manipulations while finding the eigenvalues (and maybe eigenvectors) of a Hilbert matrix.

Theory Digression

Don't enter any expressions yet.

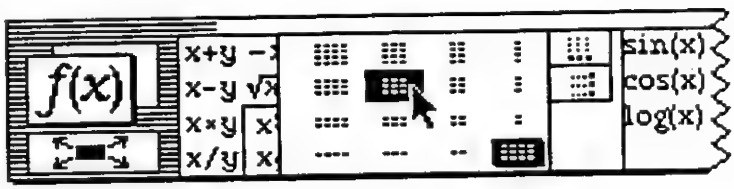
The eigenvalues and eigenvectors are the solutions of the equation $Mv = \lambda v$ where v is a vector, M is a matrix, and λ is a scalar. Given M, we will first find the eigenvalues λ and then find the eigenvector v which accompanies each eigenvalue λ. To find λ, we must solve $|M - \lambda| = 0$.

Given a matrix M, there are certain vectors v which, when transformed by M, remain pointing in the same direction (although the length may change). These are called the eigenvectors (or characteristic vectors) of the matrix. The length factor of change is the eigenvalue (characteristic value) of the matrix.

Also notice that the length of v is arbitrary. If a given vector v_1 is an eigenvector, then so is $-5v_1$. This will become important.

Creating a Matrix

Our first step is to enter a Hilbert matrix. A Hilbert matrix has a regular pattern to its components as follows. Type "M=" and choose the 3×3 matrix button from the function palette's pop-up matrices sub-palette.



...which gives you:

$$\square\, M = \begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix}$$

Select the entire matrix by dragging from the upper-left cell down to the lower-right cell. All the question marks should highlight as a single block. Now choose **Select In** from the **Edit** menu, or press Command-E. Instead of one big selection, you get nine individual selections.

Next, type "1/1+" and all the selections are entered in unison.

Pull down the **Input** menu and choose **Row Number** and then type "+". Choose **Column Number** from the same menu. You now have a Hilbert matrix.

$$\square M = \begin{pmatrix} \frac{1}{1+1+1} & \frac{1}{1+1+2} & \frac{1}{1+1+3} \\ \frac{1}{1+2+1} & \frac{1}{1+2+2} & \frac{1}{1+2+3} \\ \frac{1}{1+3+1} & \frac{1}{1+3+2} & \frac{1}{1+3+3} \end{pmatrix}$$

Select the whole equation by dragging over it or clicking on its equal sign. Hold down the Option *and* Command keys simultaneously as you double click on the equation. After you declare *M* as an M-Linear op, Theorist gives you:

$$\square M = \begin{pmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix}$$

Finding Eigen-values

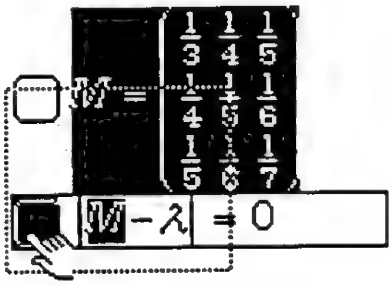
You will recall that in order to find the eigenvalues, we must solve the equation $|M-\lambda|=0$. Press Return and type the following, being careful with the different "(" and "|":

(|(M- Option-L)=0

Press Command-space and declare λ as a variable. Theorist removes the extraneous parentheses and brackets around the expression, leaving you with:

$$\square\ |M-\lambda| = 0$$

Substitute the simplified '$M =$' matrix equation into the M of this new assumption:



...so you get:

$$\triangle \left| \begin{pmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix} - \lambda \right| = 0$$

Expand it a couple of times:

$$\triangle -\frac{1}{4}\left(\frac{1}{4}\left[-\lambda+\frac{1}{7}\right]-\frac{1}{30}\right)+\frac{1}{5}\left(-\frac{1}{5}\left[-\lambda+\frac{1}{5}\right]+\frac{1}{24}\right)+\cdots\cdots$$

$$\triangle -\lambda^3+\frac{71}{105}\lambda^2-\frac{317}{25200}\lambda+2.6455\times10^{-6}=0$$

Select the polynomial (the expression on the left side of the equal sign) and Factor it. When you get a factor dialog, specify that you want a numeric answer.

$$\triangle -(\lambda-0.00021274)(\lambda-0.018926)(\lambda-0.65705)=0$$

Finding Eigen-vectors

These case theories should appear side-by-side on your screen. You can arrange them like this if you want.

Turn on the **Auto Casing** option from the **Prefs** menu. Take the first λ in the factored result and Isolate it.

$$\bigcirc\ \square\ (\lambda-0.018926)(\lambda-0.65705)\neq 0$$
$$\triangle\ \lambda = 0.00021274$$

$$\bigcirc\ \square\ (\lambda-0.018926)(\lambda-0.65705)=0$$
$$\triangle\ 0=-0$$

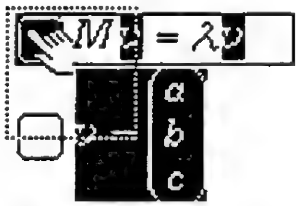
Create two new assumptions underneath these two theories. In the first new assumption, type:

M*v= Option-L ***v**

In the second, type the following:

v=(a;b;c)

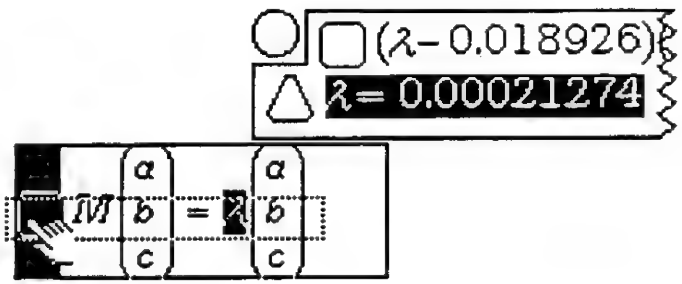
Clarify the notebook and declare v as an M-Linear op and make b a variable. (The names a and c are declared by Theorist in all new notebooks.) Substitute the whole v equation into the $Mv = \lambda v$ equation,



...which gives you:

$$\square M\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Now, from the first case theory, Substitute the equation $\lambda = 0.00021274$ into this new equation.



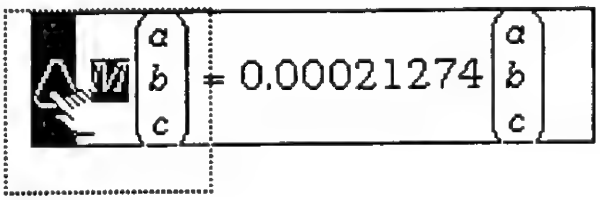
The following new conclusion appears within your case theory:

$$\triangle M\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0.00021274\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Now take the first proposition at the beginning of your notebook,

$$\square M = \begin{pmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix}$$

...and Substitute it into the recent equation in the case theory:



...which gives you:

$$\triangle \begin{pmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0.00021274 \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Select the expression to the right of the equal sign, and choose **Move Over** from the **Manipulate** menu.

$$\triangle -0.00021274 \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

Expand it and you end up with a vector like this:

$$\triangle \begin{pmatrix} 0.33312a + \frac{1}{4}b + \frac{1}{5}c \\ \frac{1}{4}a + 0.19979b + \frac{1}{6}c \\ \frac{1}{5}a + \frac{1}{6}b + 0.14264c \end{pmatrix} = 0$$

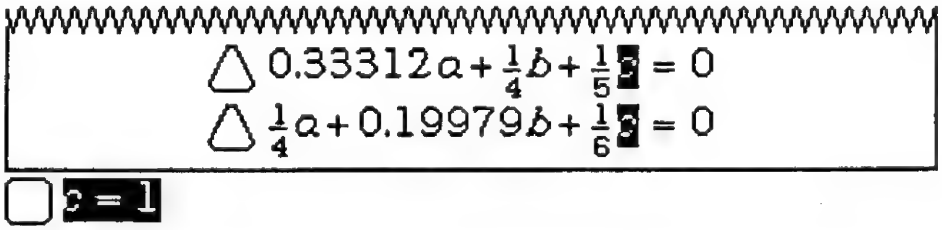
These are three equations in three unknowns, all locked into one. To separate them out, select the top sum in the matrix and Isolate it.

$$\triangle\ 0.33312a+\tfrac{1}{4}b+\tfrac{1}{5}c=0$$

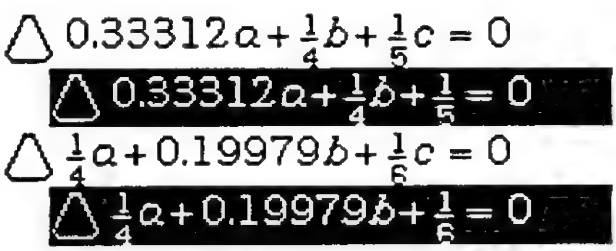
Now Isolate the second row of the matrix.

At this state of the calculation, there is a catch: the three equations are not totally independent. There are really only two good equations and the third is redundant. Therefore, your 'answer' has an extra ambiguity which translates to an extra degree of freedom. You will recall that the length of an eigenvector is arbitrary.

So what we are really doing is finding *ratios* between *a*, *b*, and *c*. We can make our work easier by just declaring $c = 1$. Create a new assumption, type "c=1", and select it. Shift-drag over the *c*'s in the last two conclusions you derived...



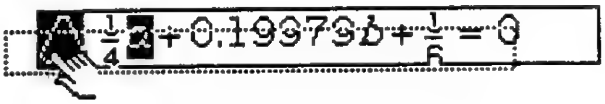
...then choose **Substitute** from the **Manipulate** menu to do a simultaneous substitution to eliminate *c*.



Now we have a straightforward two equations in two unknowns problem. Take the equation $0.33312a+\frac{1}{4}b+\frac{1}{5}=0$ and solve for *a* by isolating it. Expand the answer.

$$\triangle\ a=-0.75048b-0.60038$$

Take the resulting equation and Substitute it into *a* in the second equation.



Expand the result. Isolate *b*.

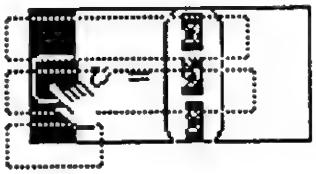
$$\triangle b = -1.3619$$

Substitute it into the *b* in the equation for *a*,

...which gives us:

$$\triangle a = 0.42169$$

Now we will substitute the answers for *a*, *b*, and *c* into our vector. You can do them all at once by Shift-clicking on the equal sign for each answer, then Command-dragging them over the proposition icon of the destination equation.



This results in the following equation.

$$\triangle v = \begin{pmatrix} 0.42169 \\ -1.3619 \\ 1 \end{pmatrix}$$

This is your eigenvector to go along with that eigenvalue.

Make a multiple selection of the following three equations.

$$\triangle v = \begin{pmatrix} 0.42169 \\ -1.3619 \\ 1 \end{pmatrix}$$

$$\square M = \begin{pmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix}$$

$$\triangle \lambda = 0.00021274$$

Substitute the selection into the $Mv = \lambda v$ equation.

$$\triangle \begin{pmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix} \begin{pmatrix} 0.42169 \\ -1.3619 \\ 1 \end{pmatrix} = 0.00021274 \begin{pmatrix} 0.42169 \\ -1.3619 \\ 1 \end{pmatrix}$$

Select the right side of the equation and choose **Move Over** from the **Manipulate** menu. The result is:

$$\triangle \begin{pmatrix} 2.4948\times10^{-21} \\ -3.1234\times10^{-20} \\ -2.0885\times10^{-20} \end{pmatrix} = 0$$

The numbers with 10^{-21} are caused by round-off error and are all effectively zero.

Exercise

By following a similar process in the other case theory, you can find the eigenvectors of the other eigenvalues.

COMPLICATED NONLINEAR SYSTEM

In a new notebook, enter the following equations:

$$xy+yz-zx=a$$
$$xy-yz+zx=b$$
$$-xy+yz+zx=c$$

(Remember that *xy* means the variables *x* and *y* are multiplied together. If you fail to enter the multiplication symbol, you get non-italicized xy, which is a single name and will cause this tutorial to fail.) Clarify the notebook and declare *b* as a variable. Make sure the **Auto Casing** option is *not* checked—that is, *off*—under the **Prefs** menu.

You need to solve for *x*, *y*, and *z*. The usual strategy is to take an equation, solve for a variable, then plug that substitution into the other equations. If you do that here you quickly end up with a mess.

With that in mind, modify your original strategy slightly: pretend that *xy*, *yz*, and *zx* are all variables, and solve for them first. This gives you another three equations in three unknowns; apply the strategy again to solve them and get the final answer.

Start by isolating *xy* from the first equation, then take that answer and Substitute it into the other two equations.

$$xy+yz-zx=a$$
$$xy=a+xz-yz$$
$$xy-yz+zx=b$$
$$a+2xz-2yz=b$$
$$-xy+yz+zx=c$$
$$-a+2yz=c$$

To make the notebook easier to follow, take the conclusions and arrange them as shown:

$$\triangle xy = a + xz - yz$$
$$\triangle a + 2xz - 2yz = b$$
$$\triangle -a + 2yz = c$$

Now, with the last two equations, you have two equations in two unknowns. In fact you are lucky here because the last equation solves immediately for *yz*. Isolate *yz* from the $-a + 2yz = c$ equation now.

$$\triangle yz = \tfrac{1}{2}(a + c)$$

Now take the *yz* equation and substitute it into the middle equation, which gives you:

$$\triangle 2xz - c = b$$

Get a solution for *xz* by isolating it:

$$\triangle xz = \tfrac{1}{2}(b + c)$$

...and substitute the *xz* equation into the $xy = a + xz - yz$ equation:

$$\triangle xy = a + \tfrac{1}{2}(b + c) - yz$$

Into that, substitute the *yz* equation and Expand the result.

$$\triangle xy = \tfrac{1}{2}a + \tfrac{1}{2}b$$

The difficult part is done. To make it pretty, Collect everything to the right of the equal sign.

$$\triangle xy = \tfrac{1}{2}(a + b)$$

Here then are the three equations in three unknowns:

$$\triangle xy = \tfrac{1}{2}(a+b)$$
$$\triangle yz = \tfrac{1}{2}(a+c)$$
$$\triangle xz = \tfrac{1}{2}(b+c)$$

(You can drag them around to position them as above for the next steps.)

Start solving for x, y, and z. Isolate x from the xy equation.

$$\triangle x = \tfrac{1}{2}\frac{a+b}{y}$$

Substitute it into the xz equation.

$$\triangle \tfrac{1}{2}\frac{(a+b)z}{y} = \tfrac{1}{2}(b+c)$$

Next, Isolate y from the yz equation:

$$\triangle y = \tfrac{1}{2}\frac{a+c}{z}$$

...and substitute it into the previous equation:

$$\triangle \frac{(a+b)z^2}{a+c} = \tfrac{1}{2}(b+c)$$

Now, solve for z by isolating it from the previous equation:

$$\triangle z = \sqrt{\tfrac{1}{2}\frac{(a+c)(b+c)}{a+b}}$$

Substitute the *z* equation into the equation for *y*.

$$\triangle y = \tfrac{1}{2}(a+c)\frac{1}{\sqrt{\frac{1}{2}\frac{(a+c)(b+c)}{a+b}}}$$

Expand the square root, then Simplify the whole thing.

$$\triangle y = \tfrac{1}{2}\frac{\sqrt{2}\sqrt{a+c}\sqrt{a+b}}{\sqrt{b+c}}$$

Take that and substitute it into the equation for *x*.

$$\triangle x = \tfrac{1}{2}\frac{\sqrt{2}\sqrt{b+c}\sqrt{a+b}}{\sqrt{a+c}}$$

If you Expand the square root in the *z* equation, you can see the symmetry:

$$\triangle x = \tfrac{1}{2}\frac{\sqrt{2}\sqrt{a+b}\sqrt{b+c}}{\sqrt{a+c}}$$

$$\triangle y = \tfrac{1}{2}\frac{\sqrt{2}\sqrt{a+c}\sqrt{a+b}}{\sqrt{b+c}}$$

$$\triangle z = \tfrac{1}{2}\frac{\sqrt{2}\sqrt{a+c}\sqrt{b+c}}{\sqrt{a+b}}$$

SOLVING NONLINEAR EQUATIONS

Previous tutorials showed you how to solve one or more equations of the following forms:

1. an equation that has one occurrence of x (just isolate x)
2. an equation of the form polynomial = 0 (factor the polynomial; gives multiple answers)
3. a set of linear equations (isolate and substitute, or make a matrix equation)
4. a set of nonlinear equations with a pattern

This chapter is concerned with "other" equations or sets of equations. In general, this problem is more difficult because there are so many kinds of problems that need to be solved. Let's start with one equation.

(Throughout this discussion, we will assume that you are trying to solve for the variable x, or for the variables x, y, z, a, b, ...)

This file is on Disk 1, in the Mathematics folder.

Open the stationery file "Standard Rules and Declarations". We will be using some of the rules and trigonometric identities defined in it.

One Equation

Enter the following:

$$\square\, x^2 = 3389$$

Isolate x to get:

$$\triangle\, x = \sqrt{3389}$$

Calculate the whole thing to get:

$$\triangle\, x = 58.215$$

It is usually best to get an algebraic solution. Not only do you get an exact answer, but you get an answer from which you can gain more insight.

Delete the two conclusions (to avoid tainting the next thing we are going to do). Turn on **Auto Casing** (or press Command-3) and Isolate again to get:

Many equations have more than one solution. Theorist can frequently give you all of the solutions, if it is a simple algebraic situation like this.

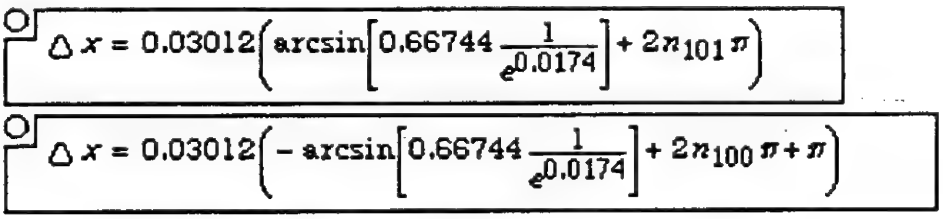
The reason that you had to delete the old answer is that Theorist feels around for some insight about the variable *x* before it goes ahead and makes multiple cases. If it finds a working definition to the tune "*x* = positive number" or "*x* > non-negative number" or similar, then it will go around assuming that *x* is positive and that the negative case is not applicable, so it will only generate one case, even if Auto Casing is on.

Delete the assumption and case theories. Enter:

$$0.66744 = e^{0.0174}\sin(33.2x)$$

Auto Casing should be left on.

Isolate *x*. You get two cases, one for the uphill side of the sine, and one for the downhill side (which is negated and offset by π).



The two constants n_{100} and n_{101} are arbitrary integers. In other words, each case represents a countable infinity of solutions. (An arbitrary integer represents a countable infinity, because you can count them. An arbitrary real number represents an uncountable infinity of solutions.)

If your equation has more than one occurrence of *x*, try to get to one occurrence by using some algebra.

Delete the assumption and case theories. Enter the following:

$$\sin(2x) = \tan(x)$$

Theorist uses transformation rules previously set up in the Standard Rules and Declarations.

Select the equation and choose **Apply** from the **Manipulate** menu. Choose **Transform** from the **Manipulate** menu to get it to the form:

$$\triangle\ 2\cos(x)\sin(x) = \frac{\sin(x)}{\cos(x)}$$

If your equation did not transform into the equation shown above, it is probably because you are not using the "Standard Rules and Declarations" notebook.

Turn off Auto Casing so the next few steps will be easier. Move the sine on the left to the right, then move the cosine on the right to the left, so you get this:

$$\triangle\ 2(\cos[x])^2 = 1$$

Then, solve for x by Isolating:

$$\triangle\ x = \frac{1}{4}\pi$$

If you turn on Auto Casing, you will get many more solutions. For instance, $x = 0$ is a solution, and the solutions repeat, as you would expect trig functions to do. (Many other symbolic algebra programs actually fail to give all of the answers for this equation.)

In the event that your equation is not so fortunate that you can move the x's together, you can use the numerical root finder in Theorist. This situation comes up a lot in real life; the problems that you are given in math books (and tutorials like this) tend to be "easy" to solve.

Enter:

$$\triangle\ \frac{1}{8}x + 0.1 = \sin(x)$$

Our first step is to convert the equation to the form "0 = everything else," so select the left-hand side of the equation and move it over to the right:

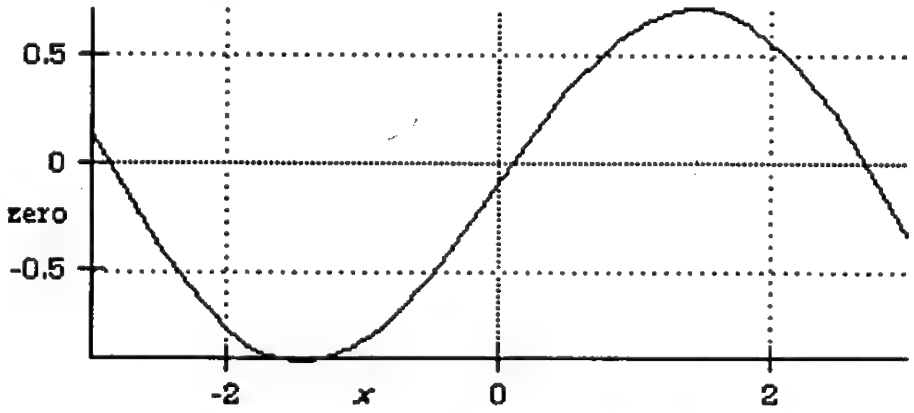
$$\triangle\ 0 = \sin(x) - \frac{1}{8}x - 0.1$$

Next, change the "0" to "zero".

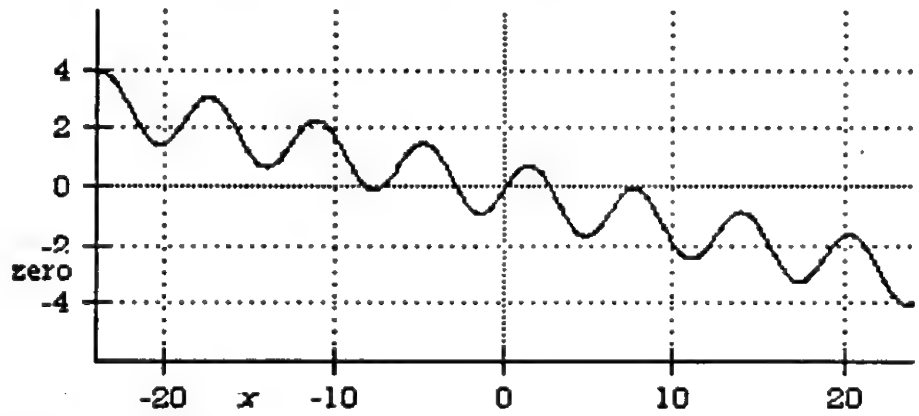
$$\square\ \text{zero} = \sin(x) - \tfrac{1}{8}x - 0.1$$

When Theorist gets around to declaring "zero" for you, call it a variable.

Now, press Command-G to see what this looks like. Our job, now, is to make zero be equal to 0, so when we look at the graph, we are interested in the zero crossings.



Well, we can see three roots here. Let's zoom out to see if there are more. Click on the rocketship icon three times and we can see that there's potential for a few more roots.

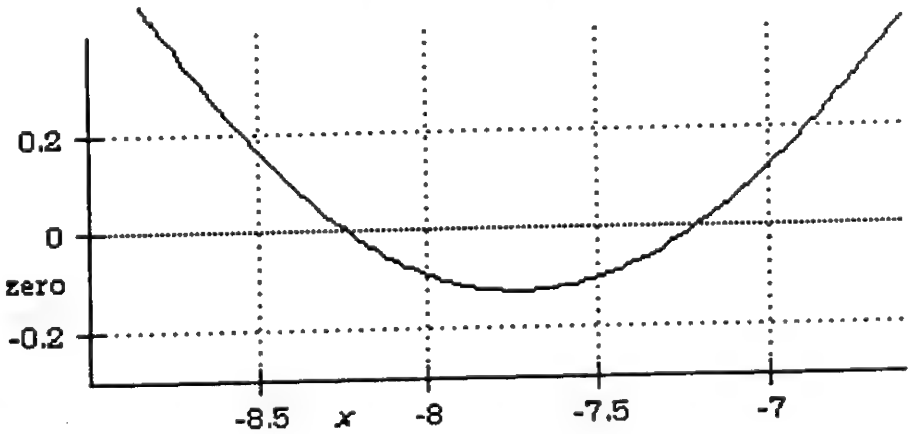


Besides the original three, the curve looks like it dips just below zero on the left, and it almost reaches above zero on the right. With the knife and rocket you can explore these and verify for yourself that at –8 or so it crosses twice, and at 8 it never quite crosses.

In the book *Numerical Recipes* (Press, Flannery, Teukolsky and Vetterling, Cambridge University Press), the authors even go so far as to supply a routine to draw a crude asterisk-plot of the function you are working with, just so you can see what you are doing and avoid entrusting your Ph.D thesis to a simple-minded blind algorithm. Using Theorist is much easier.

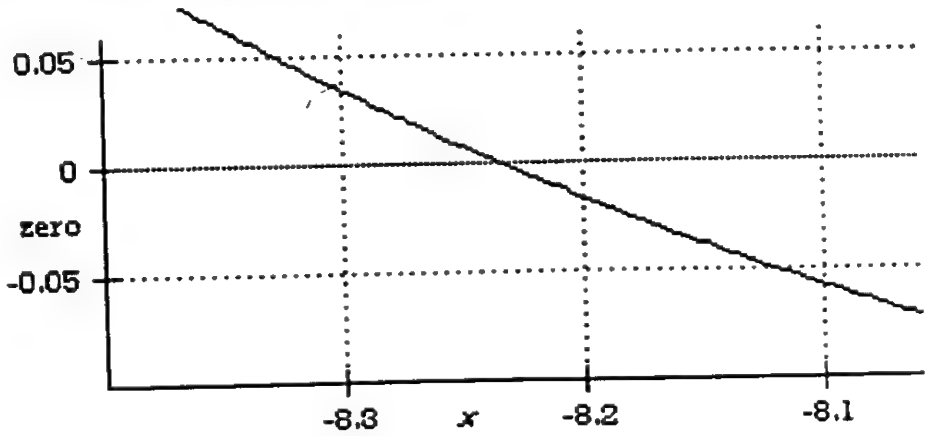
This curve demonstrates some classical pitfalls of the routine practice of having a blind computational algorithm finding one or more roots for you: there may be more than one; there may be "almost" roots (as in this case, near 8); the roots may be closely spaced and look like they are "almost" roots (as in this case, near –8), or there may be valleys that can lead your root-finding program on a wild goose chase.

We can find the left-most root by zooming in on it with the knife. Do so. From what we can see in this view, it looks to be about –8.25:

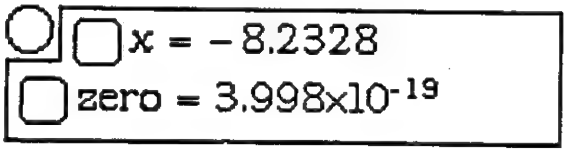


We could continue to slice with the knife so that we get more and more digits of precision. With practice, you can get about one more digit of precision with each slice. Not only is this tedious, but you will find that the graphing system breaks down at a certain point due to lack of precision.

A better approach is to use the Find Root command. Once we have isolated the zero crossing with another knife stroke,



...we can choose **Find Root** from the **Graph** menu, which calculates the root to the full 18 or 19 digits of accuracy:



The first equation gives us a value of x. The second gives us a value of "zero" at that point, which should be close to zero. (Even though only four or five digits are shown here, all 19 digits in these numbers are available by changing the display precision in the Prefs menu. The root finding algorithm attempts to calculate an answer that is accurate to the full 19 digit precision. It almost always gets at least 17 digits.)

Theorist puts the solution into a case theory because the situation where x is this value is not the whole of our work (for instance, we want to consider other roots at other values of x). You can continue any derivation you need to do with that root in its case theory.

By repeating this process, we can find all of the other roots to great precision.

When using this technique, you have the freedom to arrange the equation that you use to draw the graph a little differently. For instance, if you needed to solve $\frac{1}{x} = x$, the obvious thing is to turn it into $0 = -\frac{1}{x} + x$. By moving things around a bit first, you can arrive at $1 = x^2$ and then $0 = x^2 - 1$. Both equations have zeros at the same places, and perhaps the second one is easier to deal with because there is no singularity at $x = 0$.

Multiple Equations

Solving multiple simultaneous equations is obviously more difficult than a single equation. Some of the other tutorials have shown some tricks you can use for certain types of equations:

1. If the equations are all linear, make it a matrix equation (see the "Solving Linear Systems" tutorial).
2. If the equations have some special symmetry, you can exploit that (see the "Multiple Nonlinear Equations" tutorial).

In the usual case, though, there are few tricks you can pull. The standard algorithm for solving N equations in N unknowns is an extension of what you saw in solving two equations in two unknowns:

1. Algebraically eliminate one variable. Usually you end up solving one of the equations for a variable, "variable = other stuff", then plugging the other stuff in for the variable in all of the other equations. You now have $N-1$ equations in $N-1$ unknowns.
2. Solve the $N-1$ equation system. (This is a recursive algorithm.)
3. Plug the $N-1$ values for the other variables into your equation "variable = other stuff", and get a value for this variable.

Here's a simple demonstration of that procedure. In a new notebook, enter this system:

$$\square\ xyz = -6$$
$$\square\ x = -y\cdot 2^z$$
$$\square\ x+y+z = -4$$

Make sure the Auto Casing preference is off.

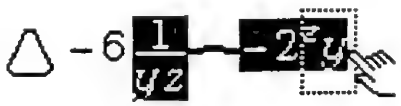
First, let's eliminate one variable, x, and one equation, the first equation. Solve for x in the first equation:

$$\triangle\ x = -6\frac{1}{yz}$$

Then, Substitute that value of x into the other two equations. Each equation turns into a new equation, and the two of them together constitute the new system of two equations in two unknowns:

$$\triangle\ -6\frac{1}{yz} = -2^z y$$
$$\triangle\ -6\frac{1}{yz}+y+z = -4$$

Drag the $\frac{1}{yz}$ from the left side over to the right side of the equation:



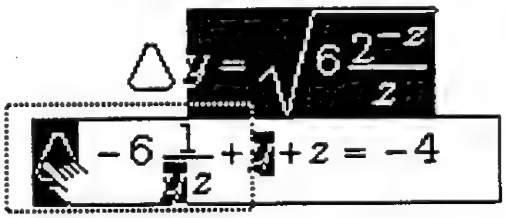
Which unites the two y's into y^2.

$$\triangle -6 = -2^z y^2 z$$

We then solve for y:

$$\triangle y = \sqrt{6\frac{2^{-z}}{z}}$$

Substitute that answer for y into the other equation,



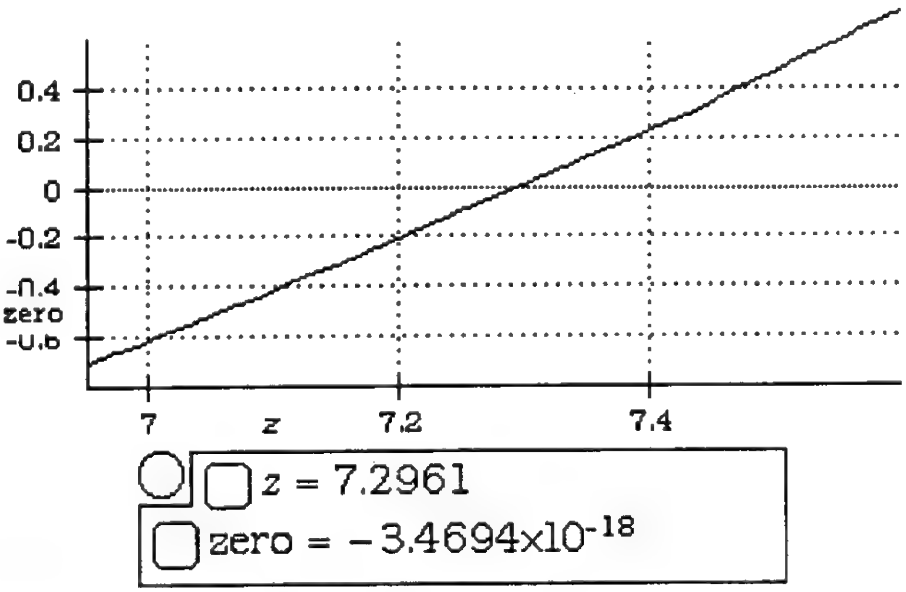
...which results in an unsightly equation:

$$\triangle -6\frac{1}{z}\frac{1}{\sqrt{6\frac{1}{2^z z}}} + \sqrt{6\frac{1}{2^z z}} + z = -4$$

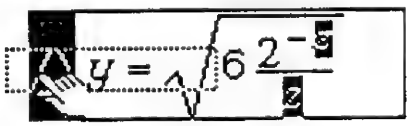
Take the left side and move it to the right side to put it in the proper form:

$$\triangle 0 = 6\frac{1}{z}\frac{1}{\sqrt{6\frac{2^{-z}}{z}}} - \sqrt{6\frac{2^{-z}}{z}} - z - 4$$

Type "zero" in place of the 0, and draw a graph. By using the rocketship and knife and then using the Find Root command, we can find a root near 7.3:



We take this value of z, and substitute it into our equation for *y*:



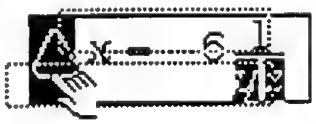
We take the answer:

$$\Delta y = \sqrt{0.82236\,\frac{1}{2^{7.2961}}}$$

...and Calculate its right side to a number:

$$\Delta y = 0.072337$$

Now that we have values for *y* and *z*, we can substitute them into the equation for *x*, the first conclusion we derived:



...which results in:

$$\triangle x = -11.368$$

Exercise

Substitute these values into the three original equations and verify that they work.

Sometimes life isn't so easy. In the last example, the whole process of finding the solution amounted to repeatedly solving equations for one variable in terms of other variables. Depending on the equations you have, this may be easy, difficult, or impossible. The algorithm is flexible in that it does not dictate which variable to eliminate with which equation; you have the freedom to choose the variable and equation that you eliminate at each step to make your work easier (or to make it possible, as the case may be).

Sometimes life still isn't so easy! Usually you get to a place where you have *M* equations in *M* unknowns and none of them can be algebraically solved for one of the variables because they are just too intertwined. If you are left with two equations in two unknowns, there is another solution: you can use the two dimensional root finder in Theorist.

Let's work with these two equations from the previous derivation:

$$\triangle -6\frac{1}{yz} = -2^z y$$

$$\triangle -6\frac{1}{yz} + y + z = -4$$

Copy them into a new notebook. We'll start by getting a graph that shows both of the equations as zero-contour plots...

Take the first of these equations, and move the $\frac{1}{yz}$ to the other side of the equation:

$$\triangle -6 = -2^z y^2 z$$

(The reason for this is that infinities bother contour plots, so we try to eliminate reciprocals as much as possible.) Next, move the –6 over to the other side, too:

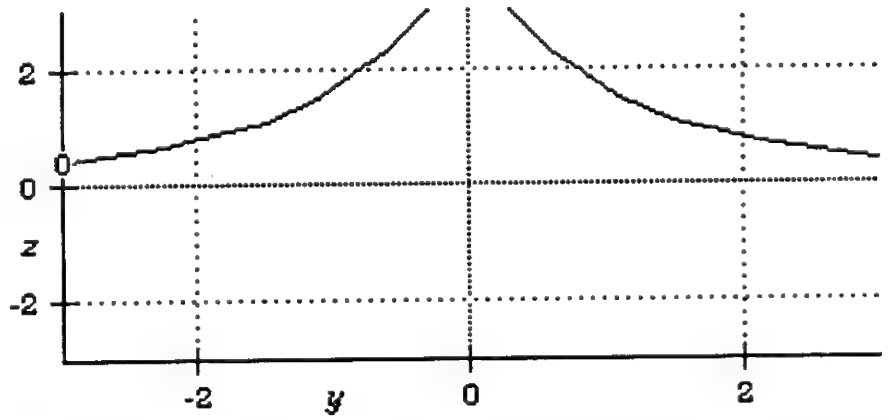
$$\triangle\ 0 = -2^z y^2 z + 6$$

Select the 0 and type "zero_1":

$$\square\ \text{zero}_1 = -2^z y^2 z + 6$$

With the cursor still in this equation, choose **Zero Contour** from the **Graph** menu's **z = f(x, y)** submenu. When the dialog comes up, the default independent variables are *z* and *y*—change this to *y* and *z*.

This makes a special contour graph showing where our expression goes to zero:



This line shows all of the (*y*, *z*) combinations where the first equation is satisfied. Of course, this is not enough; we want both equations satisfied at once.

Next, we'll add in a zero contour for the other equation. Isolate the reciprocal on one side of the equation:

$$\triangle\ 6\frac{1}{yz} = y + z + 4$$

Then, move the $\frac{1}{yz}$ to the other side, as you did with the first equation:

$$\triangle\, 6 = (y+z+4)yz$$

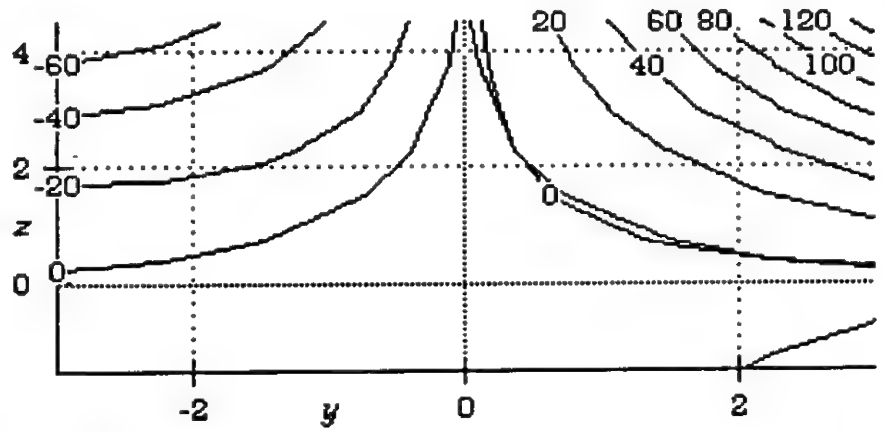
...then convert it to our favorite form:

$$\triangle\, 0 = (y+z+4)yz - 6$$

...and make the 0 into zero_2:

$$\square\, \text{zero}_2 = (y+z+4)yz - 6$$

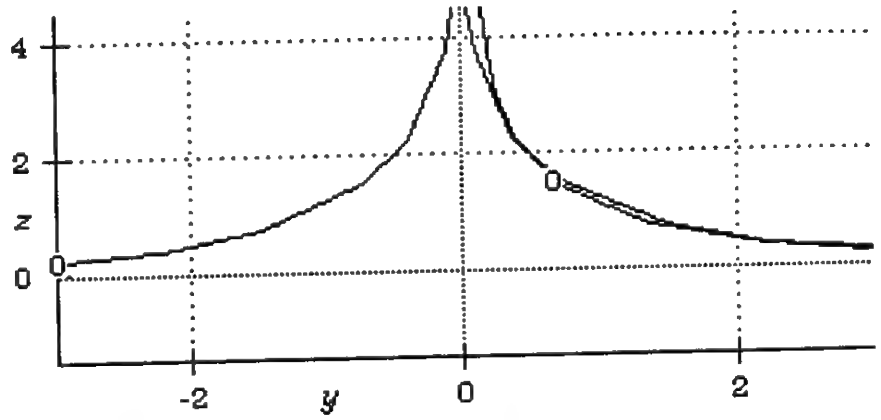
With the cursor still in this equation, choose **Add Contour Plot** from the **Graph** menu. When the dialog comes up, make sure y and z are selected in the same order again (this is to make sure that both contour plots are plotting y against y and z against z, instead of y against z). This makes the graph look like a mess:



Open the graph details and turn this added contour plot (it will be at the bottom) so it says "at zero only" instead of "normally".

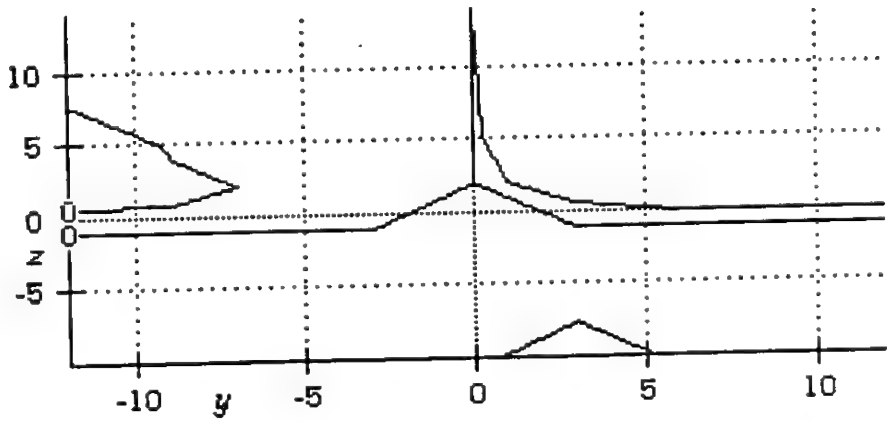
◎ Contours at (y,z) where y = left ... right and z = bottom ... top ;
at zero only spaced contours of zero_2 .

The graph becomes more simple:



Now we have the graph the way we want it: each line shows where one or the other equation is satisfied. All we have to do is find intersection points. Two directions look promising: one is above and to the right; the other becomes apparent far to the left after zooming out two steps.

A contour plot only a mother could love. The reason it is so ugly is explained below. In fact, your graph may not even look like this. You can try increasing the graph's accuracy to improve its looks.



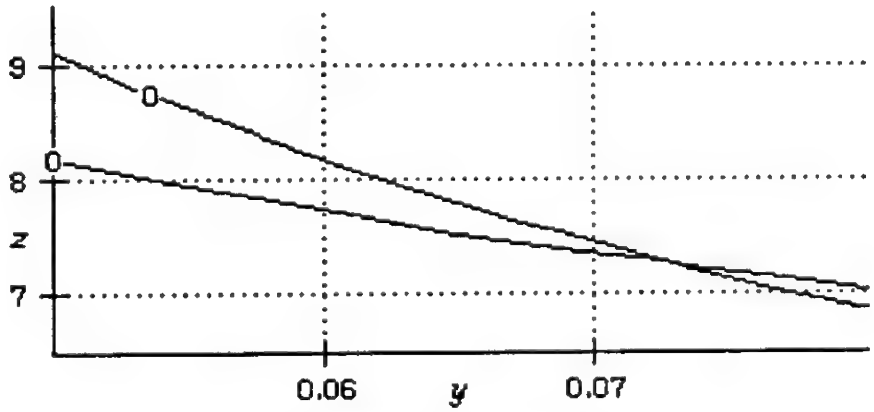
It also reveals some limitations on contour plots. The contour plot has to evaluate its function over the entire plane. This is slow. It partially compensates for this by sampling a sparse mesh of points and doing linear interpolation to find the zeros. Sometimes a linear approximation is poor, as you can see in this graph. There are two solutions to this problem:

- Increase accuracy. The default breaks up the graph area into 64 rectangles, evaluating the function at the 81 corners. Each click on the dense mesh button doubles the mesh density in each direction. A side effect is that this slows down the graphics by a factor of 4.

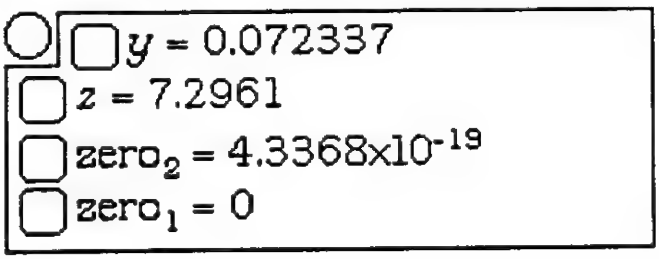
❷ Zoom in, carefully, to the points of interest to see what is really going on there. Do not zoom in too fast; you can't see where to zoom into very accurately. Two lines that pass close to each other may intersect if you zoom in closer.

Let's start by exploring the upper intersection. With the knife, zoom in on the upper middle of the graph, about a quarter or an eighth of the whole area. Keep slicing in with the knife. You may have to retreat once or twice if you try to zoom in too much in one step. What you see will be very jagged and intersection points will come and go. It's best to think of your task this way: you are eliminating the area of the graph that you know does not have the intersection in it.

Keep going until you have a clear intersection point like this:



At this point, choose **Find Root** from the **Graph** menu.



As in the case of the one-dimensional root, Theorist attempts to calculate the answer to full precision. Note that this is the same solution found the last time. By plugging these into the equation for *x*, you can find that, too.

GROUP THEORY

Group theory is a branch of mathematics concerned with multiplying abstract objects together to make other abstract objects. The objects, called "elements," can be numbers, vectors, matrices, differential operators, or rotation operators. The "multiplication" does not have to be multiplication; it can be anything that vaguely resembles multiplication. For instance, addition over integers modulus 7 constitutes a group. Group theory is not concerned so much with the mechanics inside the elements and how the "multiply" operation is done, but is concerned more with which elements multiply to get which other elements and the properties of the group of elements as a whole.

The minimum requirements for a group are that you have unique elements (at least two) and a "multiply" operation. The multiply operation must have:

1. Closure—if you multiply any two elements in the group, the result has to be in the group, too.
2. Associativity—$a(bc) = (ab)c$.
3. Identity—there must be one element like 1 where $1a = a$ for all a.
4. Inverse—for every element, there must be an inverse element in the group (which may be itself).

Notice that there is no rule about being commutative. By declaring operators to be M-Linear in Theorist, we can study group theory, because M-Linear operators are not necessarily commutative.

From these simple rules, a very interesting world opens up. For instance, there is only one unique group that has two elements in it. All other groups with two elements in it are isomorphic to it (i.e., they all work exactly the same). One representation of this group consists of the numbers 1 and –1 being multiplied together. 1 is the identity element and –1 is the other element which is its own inverse. It is like handling a coin: leaving the coin alone is the identity; turning it upside down is its own inverse.

One useful tool for thinking about groups is to make a multiplication table for it. This can easily be done in Theorist.

Enter the following expression:

$$\square\begin{pmatrix}1\\2\\3\\4\end{pmatrix}(1,2,3,4)$$

Expand it:

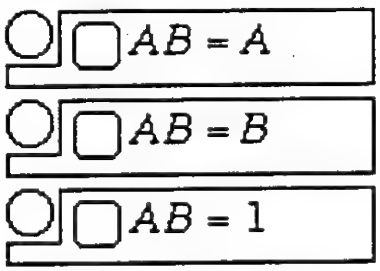
$$\triangle\begin{pmatrix}1\\2\\3\\4\end{pmatrix}(1,2,3,4)=\begin{pmatrix}1&2&3&4\\2&4&6&8\\3&6&9&12\\4&8&12&16\end{pmatrix}$$

This is a multiplication table. The top edge can be considered column headings as well as entries in the table for the row "1". Similarly, the left edge can be considered row headings as well as entries in the table for column "1". You can use this trick on any group you study in Theorist, or in fact any kinds of expressions that you want to make a multiplication table for. In this case, for instance, you can see that the numbers 1, 2, 3 and 4 do not constitute a group because it is not closed under multiplication (you get outside elements like 6 and 16).

Group of Three

Then, the question is, how many groups are there with three elements? In other words, how many different multiplication tables can you make out of the elements 1, *A* and *B*?

Create the following three case theories:



Clarify the notebook and declare *A* and *B* both as M-Linear operators. We will now try to work out as many multiplication tables as we can without contradiction.

We'll start by deriving what *AB* is. In the first case there is some element, A^{-1}, that is the inverse of *A*. We will apply that onto the front of the first equation, *AB* = *A*. Select the whole equation. Press Command-0 (that's a zero) to Apply, then the left arrow, then type

Be sure to type * between the *A* and the *B*.

"space A^-1". You get:

$$\triangle A^{-1}(AB) = A^{-1}A$$

Now, simplify it. Theorist neutralizes the *A*'s and yields:

$$\triangle B = 1$$

...which is a contradiction. If it were true, then there would be only two elements in the group, and we would not have a three element group.

There is an easier way to do this. Grab the *A* on the left side of the first equation and move it over to the other side. Theorist gives you the same result, because it internally uses the same operations to produce that result.

Delete this case theory because we know it cannot be true.

By sight, we can see the same thing will happen with the second case. If you see fit, you can go through the motions to convince yourself, this time moving *B* over. Delete this case theory also because we know it cannot be true.

So we are stuck with the last case. Given that it must be true (or else it is impossible to have a group with three elements), let's derive some other facts about it. Isolate *A* from $AB = 1$ to get:

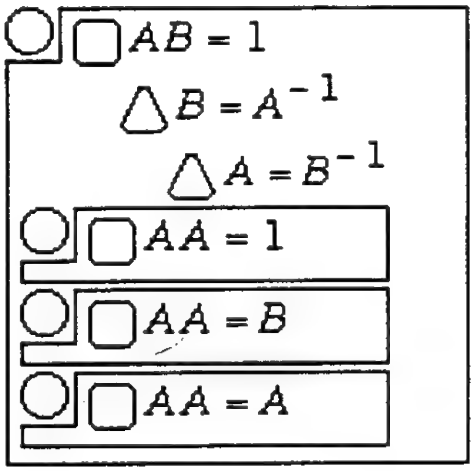
$$\triangle A = B^{-1}$$

We can also Isolate *B* to get:

$$\triangle B = A^{-1}$$

To make the three cases within the existing case theory, first create the assumptions, then surround each with the Case Theory command.

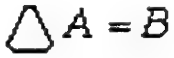
Now we have to figure out what AA, or A^2, is. In the existing case theory, make three additional cases for the three possibilities:



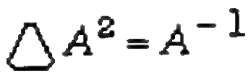
Let's start with the equation $AA = A$. Move an A over from the left to the right side. This yields a contradiction:

Therefore, the conjecture $AA = A$ is wrong. Delete its case theory.

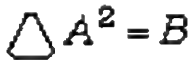
By moving over A in the $AA = 1$ trial theory, and substituting in $B = A^{-1}$ to change A^{-1} into B, we can see that it is also wrong:



Delete this theory too. Substitute $B = A^{-1}$ into $AA = B$ and simplify:



Then Substitute $B = A^{-1}$ into A^{-1}.



When you copy the equations out of the case theories and paste them elsewhere in the notebook, the conclusions change to assumptions.

Copy the remaining equations out of the case theories and paste them into the main part of the notebook:

$$\square AB = 1$$
$$\square A = B^{-1}$$
$$\square B = A^{-1}$$
$$\square AA = B$$
$$\square A^2 = A^{-1}$$
$$\square A^2 = B$$

Delete the case theories.

From these equations, we can derive equations for B^2 and BA that equate them to individual elements of the group. Start by creating this new assumption:

$$\square BA$$

Substitute $B = A^{-1}$ into it, which gives us our answer for BA:

$$\triangle BA = 1$$

Create a new assumption for BB:

$$\square BB$$

You could substitute into the right B, which results in $BB = BA^2$, which is the same answer in a different form.

Take $A^2 = B$ and substitute it into the left of the two B's, which gives you:

$$\triangle BB = A^2B$$

Take $B = A^{-1}$ and substitute it into the B of the right-hand side of the above equation. Simplify the result:

$$\triangle B^2 = A$$

Now we have enough information to make our multiplication table. Make sure that the right-hand sides of our four equations consist of just a single element.

$$\square\ A^2 = B$$
$$\triangle\ B^2 = A$$
$$\triangle\ BA = 1$$
$$\square\ AB = 1$$

Select these four equations for A^2, B^2, AB and BA *en masse* by Shift-clicking their equal signs, then choose **Transform Rule** from the **Input** menu. New transformation rule propositions appear below each equation. Change each rule's pop-up menu to "Simplify."

Upon [Simplify] transform A^2 into B .

Upon [Simplify] transform B^2 into A .

Upon [Simplify] transform BA into 1 .

Upon [Simplify] transform AB into 1 .

Next, enter this expression:

$$\square \begin{pmatrix} 1 \\ A \\ B \end{pmatrix} (1, A, B)$$

Expand it to get your multiplication table:

$$\triangle \begin{pmatrix} 1 \\ A \\ B \end{pmatrix} (1, A, B) = \begin{pmatrix} 1 & A & B \\ A & B & 1 \\ B & 1 & A \end{pmatrix}$$

This describes the one unique group that has three elements, because there are no other possibilities.

Exercise

Determine how many groups there are with four elements.

Permutation Groups

Among the most interesting kinds of groups are permutation groups. A permutation is a rearrangement of elements. For instance, if we think about a list of three numbers (1, 2, 3), there are six different ways to arrange them. We can imagine each such list as an operator, rearranging the elements of another list according to its own number order. For instance, the operator (3, 2, 1) would turn all the elements backwards, so that...

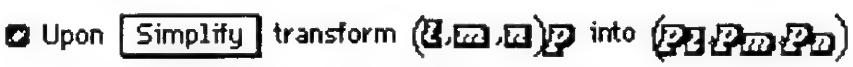
(3, 2, 1)(1, 3, 2)

...would be:

(2, 3, 1)

The identity element in this group would be (1, 2, 3). There are all sorts of little subgroups, too. For instance, the element (1, 3, 2) will simply interchange the last two elements, so it is its own inverse.

In a new notebook, we can make a rule to calculate the multiplication:



With this transformation rule, two lists (row vectors) of integers multiplied together turn into the result that you would get if you permuted them according to our guidelines. If this is applied to several lists multiplied together, the permutations of integers multiply against each other and eventually reduce down to a single list of three.

We can do this only because Theorist does not have any built-in way of processing row vectors that are multiplied. A row vector and a column vector are a different story, because matrix multiplication might happen, but that is not important for what we want to do.

Try it out by creating a new assumption and typing:

```
(3,2,1) (1,3,2)
```

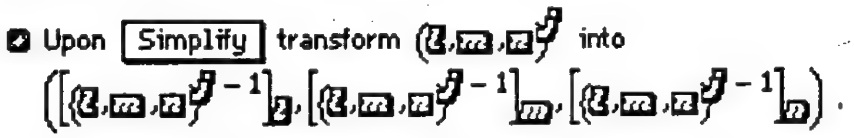
...then simplifying it:

$$\triangle(3,2,1)(1,3,2) = (2,3,1)$$

A problem occurs when you try to operate two identical lists together. Press Return and type "`(3,2,1) (3,2,1)`" then simplify it:

$$\triangle(3,2,1)(3,2,1) = (3,2,1)^2$$

In this case, Theorist is being too smart for its own good. We can work around this by including a power law which handles all powers iteratively. The easiest way to make it is to copy, paste and modify the previous rule so that our new rule is:



Now, we can go back and simplify the square we had:

$$\triangle(3,2,1)(3,2,1) = (1,2,3)$$

Not only that, but we can get totally carried away:

$$\square(3,2,1)^7(1,3,2)^3(2,1,3)^5$$
$$\triangle(3,2,1)^7(1,3,2)^3(2,1,3)^5 = (1,3,2)$$

That's six semicolons ; and six commas ,

Now, let's make a multiplication table for this. Press Return and type "(;;;;;;)(,,,,,,)".

Select the whole product of both matrices. Press Command-E twice. Type "(9,9,9)", which gives you:

The 9 is an arbitrary "filler;" *x* would be just as well.

$$\square\begin{pmatrix}[9,9,9]\\[9,9,9]\\[9,9,9]\\[9,9,9]\\[9,9,9]\\[9,9,9]\end{pmatrix}([9,9,9],[9,9,9],[9,9,9],[9,9,9],[9,9,9],[9,9,9])$$

Now, we are going to change all of those 9's to vectors. Select the six positions as shown:

$$\square\begin{pmatrix}[\blacksquare,9,9]\\[9,\blacksquare,9]\\[9,9,\blacksquare]\\[9,9,9]\\[9,9,9]\\[9,9,9]\end{pmatrix}([\blacksquare,9,9],[9,\blacksquare,9],[9,9,\blacksquare],[9,9,9],[9,9,9],[9,9,9])$$

Then type:

1 Tab 2 Tab 3

Next, select the six positions as shown:

$$\square \begin{pmatrix} [1,2,3] \\ [3,1,2] \\ [2,3,1] \\ [\blacksquare,9,9] \\ [9,\blacksquare,9] \\ [9,9,\blacksquare] \end{pmatrix} ([1,2,3],[3,1,2],[2,3,1],[\blacksquare,9,9],[9,\blacksquare,9],[9,9,\blacksquare])$$

Then, type:

3 Tab 2 Tab 1

Select the product of both matrices and expand to yield:

$$\Delta \cdots = \begin{pmatrix} [1,2,3] & [3,1,2] & [2,3,1] & [3,2,1] & [1,3,2] & [2,1,3] \\ [3,1,2] & [2,3,1] & [1,2,3] & [1,3,2] & [2,1,3] & [3,2,1] \\ [2,3,1] & [1,2,3] & [3,1,2] & [2,1,3] & [3,2,1] & [1,3,2] \\ [3,2,1] & [2,1,3] & [1,3,2] & [1,2,3] & [2,3,1] & [3,1,2] \\ [1,3,2] & [3,2,1] & [2,1,3] & [3,1,2] & [1,2,3] & [2,3,1] \\ [2,1,3] & [1,3,2] & [3,2,1] & [2,3,1] & [3,1,2] & [1,2,3] \end{pmatrix}$$

Notice that since we put the identity element in the first position, the top row serves as column headings, and the left column serves as row headings.

Looking at this table we see four subgroups in it. The first is the upper left quarter of the table, a square of nine cells, whose elements are all the set (1, 2, 3) in various stages of rotation. This is isomorphic to the group of three from the last section (because it is a group with three elements in it).

The second group is found at the intersections of the first and fourth rows and columns, with the elements (1, 2, 3) and (3, 2, 1). You can think of this group as the one that rearranges the elements backwards and forwards, or you can think of it as the one that interchanges the first and last element. This is of course isomorphic to the simple group of two discussed earlier.

Similarly, the third subgroup can be seen in the first and fifth rows and columns, and the last subgroup can be seen in the first and last rows and columns. These also interchange two elements. This illustrates a very interesting area of group theory, the study of subgroups.

These examples have been relatively simple; most people who do group theory have this permutation group memorized. Group theory is used for a wide variety of things such as Rubik's Cube, rotations in space, and in a wide variety of areas in physics concerning quantum mechanics and subatomic and sub-nuclear particles.

Calculus

Calculus

Pressing the Tab key moves the selection to the next **?**, which is for the lower limit. Type in "**-1**". Press Tab again to advance the cursor to the next **?**, which is for the upper limit. Type "**2**":

$$\square \int_{-1}^{2} (x^2) dx$$

Now select the whole expression by clicking on the integral sign, then choose **Simplify** from the **Manipulate** menu. Before the simplification is performed, you get a dialog box asking you to define *d*; click on its **Predefined** button when the dialog appears.

Theorist then proceeds with the simplification. Since Theorist's **Auto Simplify** option is on, it goes through a few steps of simplification and gives you:

$$\triangle \int_{-1}^{2} x^2 dx = 3$$

Your answer is three. Piece of cake!

Exercise

Delete the answer and turn the Auto Simplify preference off. Solve the integral, and watch the steps Theorist goes through. Simplify until the above result is reached.

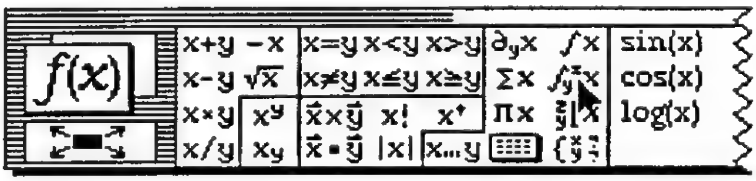
SIMPLE DEFINITE INTEGRAL

$$\int_{-1}^{2} x^2\,dx$$

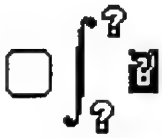
In this tutorial you learn how to enter and solve the above simple definite integral. More difficult examples follow.

Use the palette for this tutorial. If you created a new notebook for this tutorial (like you are supposed to do for all tutorials) and haven't changed it, the palette at the top of the screen is the variables palette (a lone *x* is on the left). (The function palette has *f(x)* on the left.)

Make the function palette active, then click on the definite integral:

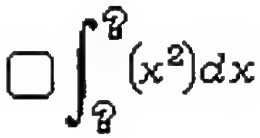


...which makes:



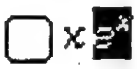
Shift-6 for "^"
Shift-8 for "*"

The highlighted question mark is the current selection, or insertion point. Type in the expression to be integrated, "`(x^2)`". To enter *dx*, type "`d*x`". You get the following:



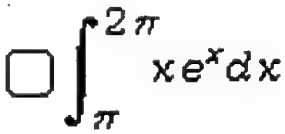
This tutorial uses the palette extensively; you may use the keyboard if you wish.

Click these buttons in succession: x, $x \bullet y$, e, x^y, x, and $\rightarrow$. So far you should have:

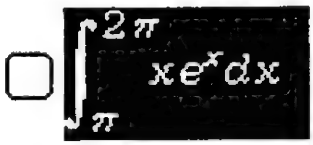


Click $x \bullet y$, d, $x \bullet y$, x, and $\rightarrow$. On your screen you see:

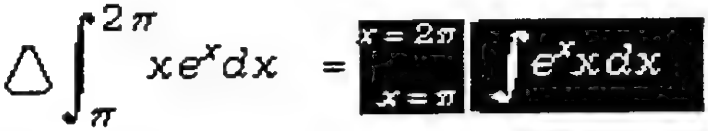
Back to the function palette... Click $\int_y^x x$ and π. Select the upper limit (the ?) and type "2" from the keyboard. Click $x \bullet y$ and π. Click outside the expression (to get rid of the selection) and it looks like this:



Now that the integral is entered, you are ready to do some work on it. Select the whole expression by clicking on the integral sign:



Simplify.



Theorist took the limits off the integral, but could not integrate it. We can do it by integrating by parts. First, we have to decide which part to integrate and which part to differentiate.

e^x stays the same if we integrate it or differentiate it. x gets more complicated if we integrate it, but it turns into 1 if we differentiate it. We should select the part to integrate, so we want to select e^x and dx.

INTEGRATION BY PARTS

$$\int u\,dv = uv - \int v\,du$$

The integration by parts rule.

Integration by parts is an integration technique which is useful if your integrand consists of two parts multiplied together, one part that you know how to integrate and one that you know how to differentiate. Since the end result includes another integral, it does not always solve your integral.

Theory Digression

To do an integration by parts with Theorist, decide which part you want to differentiate. Choose the part you want to integrate, and select **Int. By Parts** from the **Manipulate** menu. (If it was called "differentiate by parts," you would have to select the other part.)

Do not type or click anything yet.

This tutorial uses two new, undeclared names: *d* and *e*. So before you start the derivation process, declare these names. (It is not always necessary to declare a name before you use it, but in this case we are using the names *d* and *e* up on the variables palette, which contains only previously declared names.)

Now you can start.

Type "`d*e`" into a blank assumption. Then choose **Clarify** from the **Notebook** menu, or press Command-space. A dialog box asks you to declare *d*—do so by clicking **PreDefined**. Another dialog appears, this one asks you to declare *e*—do so by clicking **PreDefined** once more. The names *d* and *e* are now declared as a differential operator and a natural log base, respectively.

Delete the *d e* proposition because you are done with it, and to avoid any possible confusion. Press Return for a new proposition. The function palette, *f(x)*, switches to the variables palette, *x*. Since the names are now declared, *d* and *e* appear with the other declared names such as π and *x*, on the variables palette.

Now we begin entering the expression, which we do almost entirely from the palette; you only have to press one key from the keyboard. (Of course, you may enter the equation from the keyboard.)

Simplify the expression.

$$\triangle \int_{\pi}^{2\pi} x e^x dx = \left[\int e^x x\, dx \right]_{x=\pi}^{x=2\pi}$$

Select the *dx* alone, and Simplify. You get a slightly different form:

$$\triangle \int_{\pi}^{2\pi} x e^x dx = \left[\int e^x x (dx) \right]_{x=\pi}^{x=2\pi}$$

Note the parentheses: now, this is the product of three things; the last thing is a *dx*. Now, select the e^x and the *dx* simultaneously, and Integrate by Parts:

$$\triangle \cdots\cdots = \left[-e^x - c_{103} x + \left(e^x + c_{103}\right) x - c_{105} + c_{104} \right]_{x=\pi}^{x=2\pi}$$

Expand the right side to get the final answer.

$$\triangle \int_{\pi}^{2\pi} x e^x dx = -e^{2\pi} + e^{\pi} - e^{\pi}\pi + 2e^{2\pi}\pi$$

Because they are not next to each other, grab the e^x and move it to between the x and the dx, like this:

$$\Delta \int_{\pi}^{2\pi} x e^x dx = \left[\int e^x x \, dx \right]_{x=\pi}^{x=2\pi}$$

...which gives us:

$$\Delta \int_{\pi}^{2\pi} x e^x dx = \left[\int x e^x dx \right]_{x=\pi}^{x=2\pi}$$

Before you do the next step, turn on the **Auto Casing** option under the **Prefs** menu.

Now select $e^x dx$ and choose **Int. By Parts** from the manipulate menu. (When you get a dialog asking you to declare ln, make it predefined.)

$$\Delta \cdots\cdots = \left[-e^x - c_{100}x + (e^x + c_{100})x - c_{102} + c_{101} \right]_{x=\pi}^{x=2\pi}$$

The expression to the left of the equal sign has been collapsed for space reasons. Your screen should show the entire equation.

With your answer, the arbitrary constant, c, may have subscripted numbers different from those shown above; this is fine.

Select the expression to the right of the equal sign and Expand:

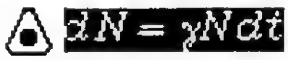
$$\Delta \int_{\pi}^{2\pi} x e^x dx = -e^{2\pi} + e^{\pi} + 2e^{2\pi}\pi - e^{\pi}\pi$$

...which is the final answer.

An alternate way to do the integration by parts is to select both the e^x and the dx simultaneously and choose **Integrate by Parts**. Unfortunately, Theorist does not allow you do that because selecting the dx means selecting the product of all four things. Instead, let's use a different method...

Copy the original expression (the integral) into a new notebook, making sure that the Auto Casing preference option is on.

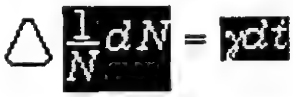
Now we are ready to start working on the equation. Select *dt* and choose **Move Over** from the **Manipulate** menu. Notice how the conclusion generated is a working statement.



From the expression on the right side of the equal sign, select *N*. Use Move Over, but this time by Command-dragging rather than using the menu:

This results in:

With the equation selected as above, choose **Apply** from the **Manipulate** menu. This has the effect of selecting the expressions on both sides of the equal sign, but not the whole equation with the equal sign:



Next we want to integrate both sides of the equation. Click on $\int x$ from the function palette.

From the **Prefs** menu, turn on the **Auto Casing** option. This puts Theorist in strict derivation mode, in which it pays close attention to the subtle details and rules of mathematics. We are using it now to generate some special constants, explained shortly hereafter.

Click on the equal sign and choose **Simplify** from the **Manipulate** menu. (If you get a name declaration dialog for "ln", make it Predefined.)

DIFFERENTIAL EQUATION

In this tutorial we solve a simple differential equation:

$$\frac{dN}{dt} = \gamma N$$

This type of equation arises in population modeling. Once you master the technique you can attempt more challenging problems involving differential equations.

Start off fresh with a new notebook. Set your preferences as shown at left. (Set the palette and keyboard entry preferences to suit your desired method of equation entry.)

Enter the above equation, using your favorite entry method. If you need help, just type the following:

`(d*N) / (d*t)` Esc = Option-G `*N`

...Which gives you on-screen (effectively if not literally):

$$\square \frac{dN}{dt} = \gamma N$$

During this clarification, if your equation has the unnecessary parentheses, Theorist eliminates them.

Verify γ is a constant; if it is not, this will not work.

Our next step is to choose **Clarify** from the **Notebook** menu. Theorist shows a dialog for each undeclared name. Declare them using the following table as a guide.

Name	Declaration
d	Predefined
N	User-Defined Variable
t	User-Defined Variable
γ	User-Defined Constant

Basically, all you have to do after issuing the Clarify command is to press Return three times, then set the pop-up menu to **Constant** and press Return again.

To solve for N, select it and choose **Isolate** from the **Manipulate** menu. (If you get a name declaration dialog for "exp", make it Predefined.) The answer:

The constants c_{101} and c_{100} were generated by Theorist when you integrated the differential equation. They are arbitrary; any value you choose still works as a solution to the original differential equation. (See a calculus or differential equations book for more details.) Since they both are arbitrary, and only the difference is important, you may safely delete $-c_{100}$ without causing any problems.

If this were a real-world problem, you could solve for c_{101} from your boundary conditions.

Exercise

For the above equation, assume you need a solution for the case:

$$N = 645.3 \text{ when } t = 1991$$

Also assume $\gamma = 0.03$. Find a value for c_{101}. This is a model for the population of Africa, in millions of people.

Type "x, y" in the question mark. (The parentheses appear automatically.)

the variables (x,y) are independent of [each other]

Now, one Simplify command integrates the integral:

$$\int_0^1\left(\int_0^y \sin[x+y]\,dx\right)dy = -\frac{1}{2}\sin(2)+\sin(1)$$

You can also calculate multiple integrals the way you can calculate single integrals, with the Calculate command.

$$\int_0^1\left(\int_0^y \sin[x+y]\,dx\right)dy = 0.38682$$

Exercise

With three manipulations and no copying or pasting, find the difference between the numerical answer and the algebraic (exact) answer.

Multiple Integral Comments and Techniques

Calculating multiple integrals can take a long time. To calculate a multiple integral, Theorist has to evaluate the integrand many times for different values of the variable of integration. It strategically chooses values of the variable to maximize accuracy while minimizing the number of times it must evaluate the integrand, searching for violent fluctuations that require more evaluations to carve out at a high level of accuracy. For most situations, it generates about nine digits of precision. (That is relative to typical values of the integrand. If you integrate sin(*x*) over a cycle, you get a result that is about 10^{-10}, not 0, which is the correct answer. That is because the typical values of the integrand are about ±0.5.) At best (low-order polynomials), it evaluates the integrand more than two dozen times. For difficult situations, it could evaluate the integrand hundreds or even thousands of times. In extreme cases it may take close to a million or more evaluations.

When you do a double integral, that number is squared. Instead of undergoing about thirty to 3,000 evaluations, it may undergo 1,000 to ten million evaluations. For a quadruple integral, we're talking about one million to 100 *trillion* evaluations! The smoothness makes a major difference in the execution speed. Therefore, you should avoid calculating multiple integrals in the case of integrands with rapidly changing or

MULTIPLE INTEGRALS

Multiple integrals are simply integrals that are nested inside of each other. Enter the following integral:

$$\int_0^1 \left(\int_0^y \sin[x+y] dx \right) dy$$

Note that the inner limits can depend upon the outer variables, but not the other way around. Really, the limits define a piece of surface (or space, if a triple integral, or a 17 dimensional region if a 17-ple integral) that the integral is to be taken over. The integral above is fairly simple, just taking it over a right triangular area 1 unit on each side. In more complicated situations, the upper and lower limits must trace out the upper and lower edge of the surface to take the integral over. Integration surfaces with holes, broken pieces or what not will require more work (perhaps using more than one integral for different pieces, or multiplying in a function that is zero or one for different areas).

Simplify and notice that it does not go very far.

$$\triangle \cdots\cdots = \begin{matrix} y=1 \\ y=0 \end{matrix} \left[\int (dy) \begin{matrix} x=y \\ x=0 \end{matrix} \left[\int \sin[x+y] dx \right. \right.$$

The integrals with limits turn into integrals enclosed in Evaluate At ops, but the integrals themselves do not get processed. The inner integral does not move because it involves a variable, y, with an uncertain relationship to the variable of integration, x. As far as it knows, y could be equal to x or some function of x, or vice versa.

There is one important trick to doing multiple integrals, and indeed to doing any algebraic work in two dimensions. You must use an independence declaration. Choose **Independence Decl.** from the **Input** menu. It makes this independence declaration:

the variables ? are independent of [each other]

Note that it does not simplify because sin(sin[x]) has no integral in terms of familiar functions. We can try to split it up using trig identities for addition, but that leads nowhere. (Try it.)

The critical thing about taking a multiple dimensional Taylor series is to make the variables independent. If you don't, all of the derivatives with respect to *x* get snagged on every occurrence of *y*, and it all turns into a mess. Fortunately, we already have the correct independence declaration lying around because, for our purposes, they are independent.

Select the integrand (without the *dx*) and choose **Taylor Series** from the **Manipulate** menu. At the dialog, make sure *x* is selected in the left scroll box and press Return. (Three terms is fine for now.) When that is completed, choose **Taylor Series** again, keeping the same selection. This time, at the dialog, select *y* in the left scroll box. You end up with an integral of a polynomial:

$$\int_0^1\left(\int_0^y\left[-\tfrac{1}{3}x^3+x+x^2y^3-\tfrac{1}{3}y^3+x^3y^2-xy^2-x^2y+y\right]dx\right.$$

If you expand the whole integral, it reduces to a simple answer:

$$\Delta\int_0^1\left(\int_0^y \sin[\sin(x+y)]\,dx\right.\Big)dy \;=\tfrac{1}{3}$$

How close is this? Let's calculate the answer (it does not take very long on a fast machine):

$$\Delta\int_0^1\left(\int_0^y \sin[\sin(x+y)]\,dx\right.\Big)dy \;=0.34165$$

This answer is good to about five digits, showing how poor the Taylor series is, especially since the Taylor series took more work and more time. In fact, the numerical integration algorithm is actually very good and fast for smooth functions. It works, but is slower for violent functions. Taylor series only really works for smooth functions, in general the same kind that the numerical integration algorithm works well for. A better approximation for this particular function may have been a Fourier series.

difficult parts. If you are not sure whether your integral is "difficult," be prepared to find out.

In reality, Theorist relaxes the accuracy thresholds for the deeper levels in order to reduce the enormous execution times for multiple integrals, although you still get a reasonably accurate answer (about four or five digits for a double integral, less for larger integrals).

Here are some techniques to try to avoid calculating a multiple integral:

1. Evaluate it symbolically. Even if the whole thing can't be evaluated symbolically, sometimes integration along one or more variables can be done symbolically. You may want to reshuffle the integral, making the inside one the outside one and vice versa; you probably must do something funny with the limits. Be careful.

2. Separate the integrand into parts that are independent of one or the other variable. If the two parts are added together, they are separate integrals. If they are multiplied, you can separate them into two integrals that are multiplied together, which calculates much faster.

3. Change the coordinate system. For instance this integral,

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\right)dy$$

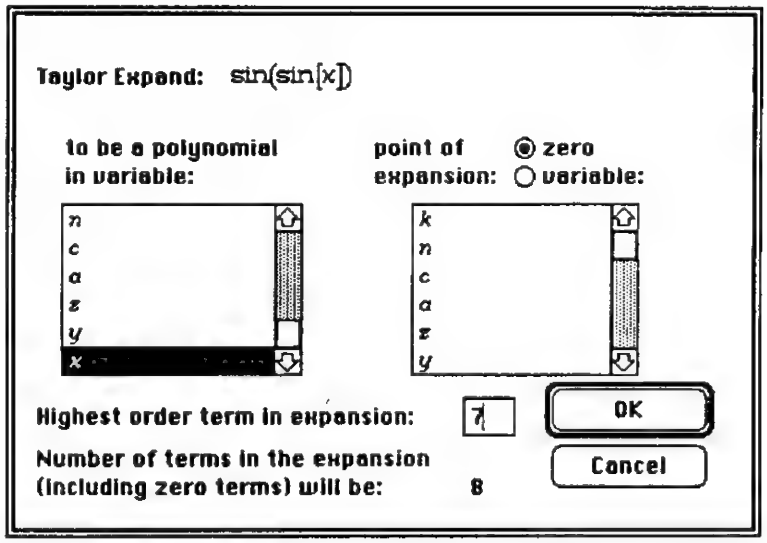
...can be evaluated by changing to polar coordinates. (Be careful with such a conversion; you must convert the differential in addition to the limits, and Theorist does not do it for you because the conversion of the limits is very subtle.) At that point, it can even be evaluated symbolically.

4. Use some sort of approximation for the integrand.

To illustrate an example of the last technique, let's modify our integral to be more difficult. Select the integrand (without the dx), $\sin(x+y)$, and click on the palette icon for $\sin(x)$:

$$\int_{0}^{1}\left(\int_{0}^{y} \sin[\sin(x+y)]\,dx\right)dy$$

Change it to 7:



Now click on the **OK** button and Theorist crunches through computations to give you the final result, with the highest order, surprisingly, seven:

$$\triangle y = -\frac{8}{315}x^7+\frac{1}{10}x^5-\frac{1}{3}x^3+x$$

To see how good the approximations are, you can graph them on the same axis. First change your equations so that the first Taylor series (up to x^3) is y', and the second Taylor series (up to x^7) is y''. Add the apostrophes to the y's (Theorist creates new assumptions), then Clarify the notebook and declare the y's as variables.

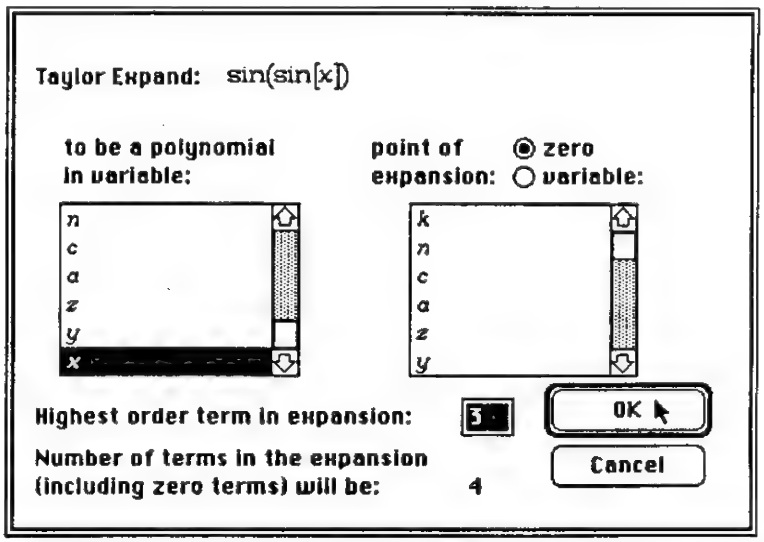
$$\square y = \sin(\sin[x])$$
$$\triangle y = -\frac{1}{3}x^3+x$$
$$\square y' = -\frac{1}{3}x^3+x$$
$$\triangle y = -\frac{8}{315}x^7+\frac{1}{10}x^5-\frac{1}{3}x^3+x$$
$$\square y'' = -\frac{8}{315}x^7+\frac{1}{10}x^5-\frac{1}{3}x^3+x$$

To graph $y = \sin(\sin(x))$, click on the equal sign in that equation and choose **Linear** from the **Graph** menu's **y = f(x)** submenu, or press Command-G. Theorist draws a graph of $\sin(\sin(x))$.

TAYLOR SERIES

This tutorial demonstrates a Taylor series approximation of a smooth function.

In a new notebook, type "`y=sin(sin(x))`". Select the right-hand side of the equation and choose **Taylor Series** from the **Manipulate** menu and the following dialog appears on the screen:



Click on the **OK** button and Theorist gives you the answer:

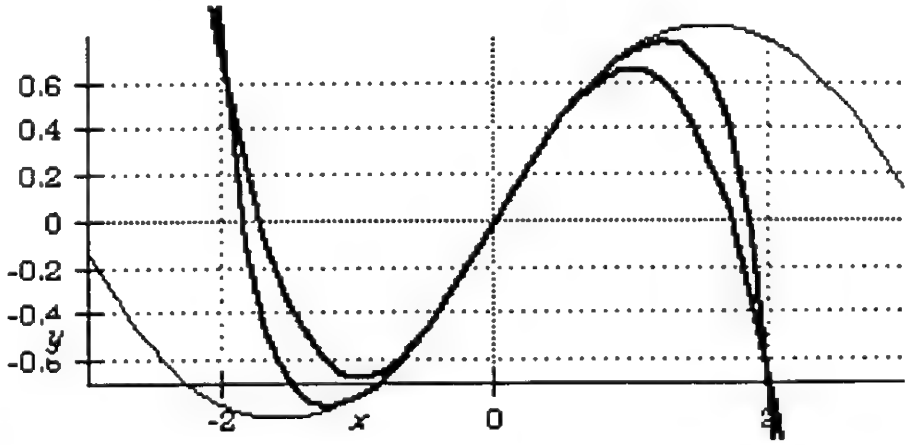
$$\triangle y = -\frac{1}{3}x^3 + x$$

As you can see, the expansion has two terms and not four as the dialog said. That's because the other two terms vanish at the expansion point, $x = 0$.

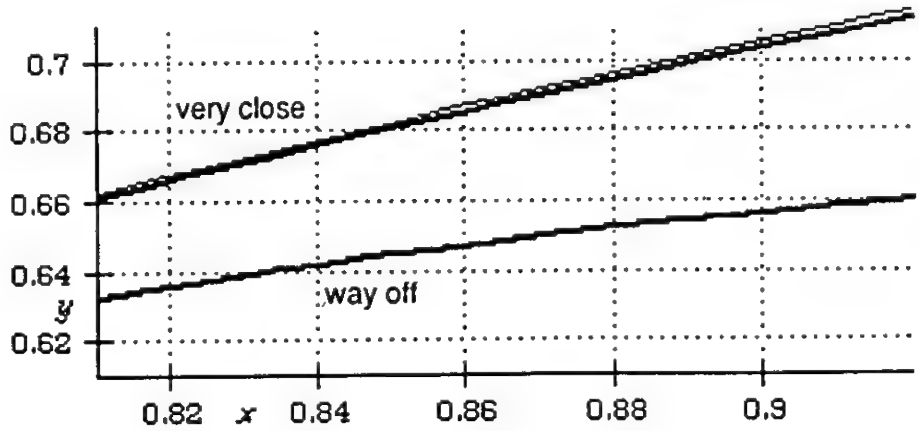
Select sin(sin(x)) from the first equation and do another Taylor Series command, but don't click **OK** yet. For a better approximation you need more terms in the expansion, which you can get by changing the number in the **Highest order term** box in the dialog.

To add a plot of: $y' = -\frac{1}{3}x^3 + x$

...select the equation and choose **Add Line Plot** from the **Graph** menu. (If Theorist gives you another dialog, click **OK**.) After the graph appears on the screen add the line plot of the y'' equation.



The thin line represents $y = \sin(\sin(x))$. The thick lines stand for the approximations which look good near $x = 0$, but get worse as you move away. From here, it looks like we did not gain very much for our two extra terms, but if you zoom in with the knife, you can see how much better the longer approximation is.



Of course, you need an expression for θ to substitute into your equation, so isolate θ:

$$\triangle\ \theta = \arcsin(x)$$

Now take this last equation and substitute it into your integral,

$$\int \sqrt{a+b(\sin[\theta])^2}\cos(\theta)\,d\theta$$

Most of the rest of this derivation takes place on the right hand side of this equation, with the left integral remaining unchanged.

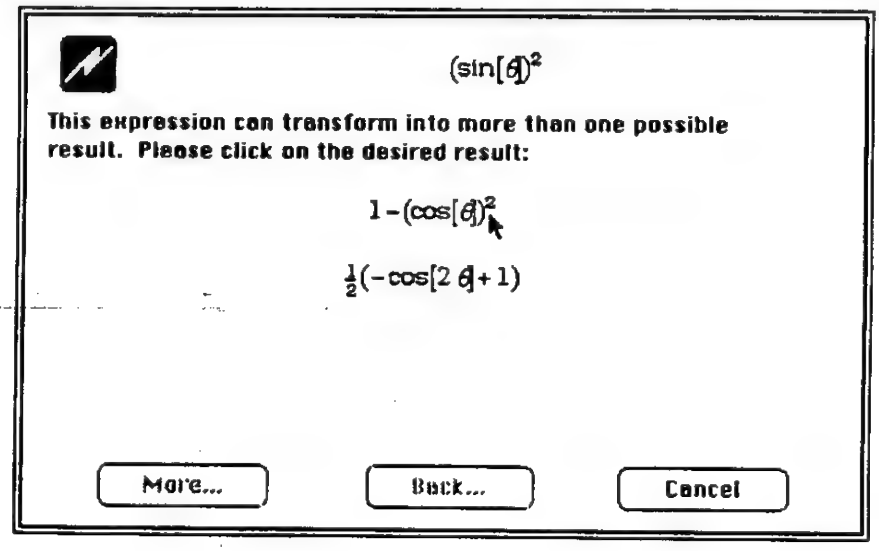
...which results in:

$$\triangle\ \cdots\cdots = \int \frac{\sqrt{bx^2+a}\cos(\arcsin[x])}{\sqrt{-x^2+1}}\,dx$$

Well, this did not end up the way we wanted it to. The expression cos(arcsin[x]) could probably be turned into that square root, except there might be a minus sign somewhere. To figure this out, we need the identity that transforms sines and cosines into each other. Enter:

$$\square\ (\sin\theta)^2$$

Select it and do a Transform manipulation. Since the Standard Rules & Declarations has two different transformation rules for this expression, Theorist presents a dialog, asking you which result you want:



COMPLICATED INTEGRAL

$$\int \sqrt{a + b(\sin[\theta])^2}\cos(\theta)\,d\theta$$

In this tutorial, we solve the above small but complicated integral. In the process, we see how to use substitution in integration, and we also use the "Standard Rules & Declarations" file.

Rumor has it that even some of the high-end symbolic algebra packages have trouble with this one. Using high-level human intelligence and low-level computer algebra, we can solve it.

Open the file "Standard Rules & Declarations", which is in the mathematics folder on your Theorist disk. This is a somewhat long file (about four pages) with many transformation rules and declarations that are handy while doing this integral, and while doing derivations in general. For instance, it has declarations for all of the trig, arc trig, and hyperbolic functions.

Since Standard Rules & Declarations is a stationery file, it opens as an untitled notebook. Save the notebook as "Integral". Save it often as you do this tutorial; it is long and involved, and you may want to take a break somewhere along the line. Saving often also decreases the risk of work lost due to system crashes. Of course, Theorist never crashes. Almost never.

At any point in this tutorial, if Theorist asks you to declare a name, press Return to accept the default declaration.

Enter this expression.

$$\square\ \int \sqrt{a+b(\sin[\theta])^2}\cos(\theta)d\,\theta$$

You can see by simplifying that it does not solve immediately. But anybody who does frequent integration can see there is a substitution just dying to be done. That cos(θ) looks like the derivative of that sin(θ), so lets try saying that x is sin(θ).

$$\square\ x = \sin(\theta)$$

Of course, you need an expression for θ to substitute into your equation, so isolate θ:

$$\triangle\ \theta = \arcsin(x)$$

Now take this last equation and substitute it into your integral,

$$\int \sqrt{a+b(\sin[\theta])^2}\cos(\theta)\,d\theta$$

Most of the rest of this derivation takes place on the right hand side of this equation, with the left integral remaining unchanged.

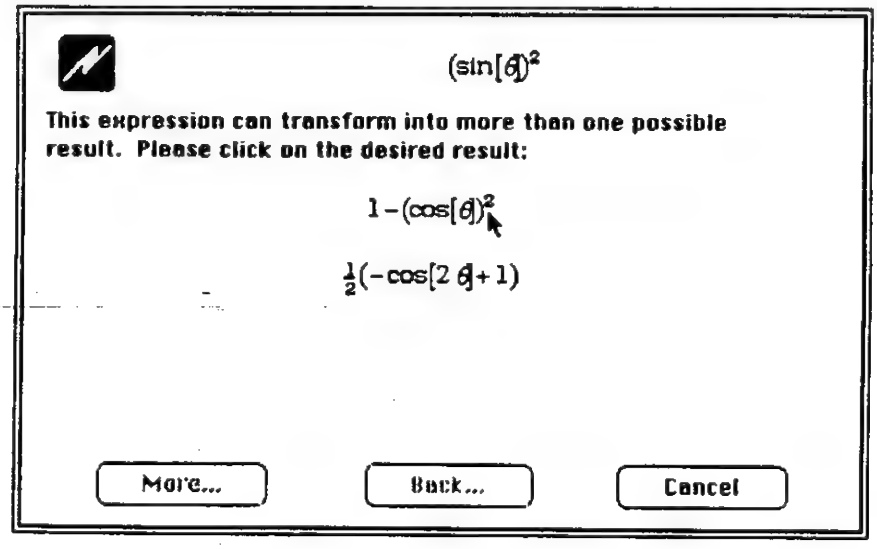
...which results in:

$$\triangle \cdots\cdots = \int \frac{\sqrt{bx^2+a}\cos(\arcsin[x])}{\sqrt{-x^2+1}}\,dx$$

Well, this did not end up the way we wanted it to. The expression cos(arcsin[x]) could probably be turned into that square root, except there might be a minus sign somewhere. To figure this out, we need the identity that transforms sines and cosines into each other. Enter:

$$\square\ (\sin\theta)^2$$

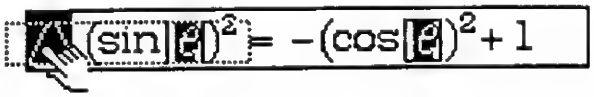
Select it and do a Transform manipulation. Since the Standard Rules & Declarations has two different transformation rules for this expression, Theorist presents a dialog, asking you which result you want:



Click on the first result shown, which produces:

$$\triangle (\sin[\theta])^2 = -(\cos[\theta])^2 + 1$$

Now, take the equation $\theta = \arcsin x$ and substitute it in to the new transformation result:



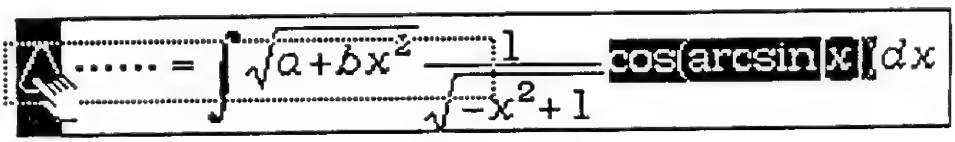
The result Auto Simplifies so you get:

$$\triangle x^2 = -(\cos[\arcsin\{x\}])^2 + 1$$

We've got it! Now, all we have to do is isolate the cos(arcsin[x]) part:

$$\triangle \cos(\arcsin[x]) = \sqrt{-x^2 + 1}$$

Substitute this into the integral:



...so you get:

$$\triangle \cdots\cdots = \int \sqrt{bx^2 + a}\, dx$$

Simplify does not crack this. All of this substitution has not gone to waste; we must go even further.

Whenever you get an integral like this, with a square root like this, a bell should go off in your head to use some sort of trig or hyperbolic function as a substitution. The idea is to get the $\sqrt{a + bx^2}$ to dissolve into something simple.

Keep the following table handy:

If your integral has this	use one of these substitutions for x
$\sqrt{x^2+1}$	tan z, sinh z, cot z, csch z
$\sqrt{1-x^2}$	sin z, sech z, cos z, tanh z
$\sqrt{x^2-1}$	sec z, cosh z, csc z, coth z

These are chosen because the functions on the right are transformed by the expressions on the left by way of trig or hyperbolic identities. Some of these substitutions work equally well (or equally poorly) in the same situations. Try a trig function first, and if that does not work, try a hyperbolic.

In this situation, we start with tangent. Unfortunately, the constants a and b complicate the situation.

Enter the substitution: $bx^2 = a(\tan[z])^2$ and isolate x, then Expand the result so you get:

$$\triangle x = \frac{\sqrt{a}\tan(z)}{\sqrt{b}}$$

This should make it all dissolve. Substitute it into the integral:

$$\triangle \cdots\cdots = \int \sqrt{b\,x^2 + a}\; dx$$

...so, after expanding, you get:

$$\triangle \cdots\cdots = \frac{\sqrt{a}\left(\int \sqrt{a\,[\tan(z)]^2 + a}\;[\sec(z)]^2\, dz\right)}{\sqrt{b}}$$

The square root can be cracked by selecting the insides,

$$\triangle \cdots\cdots = \frac{\sqrt{a}\left(\int \sqrt{a\,[\tan(z)]^2 + a}\;[\sec(z)]^2\, dz\right)}{\sqrt{b}}$$

...and doing a Collect, which gives you:

$$\Delta\ldots\ldots = \frac{\sqrt{a}\left(\int \sqrt{a\left[(\tan(z))^2+1\right]}\,[\sec(z)]^2\,dz\right)}{\sqrt{b}}$$

Select the $(\tan[z])^2$ and do a Transform manipulation so you get:

$$\Delta\ldots\ldots = \frac{\sqrt{a}\left(\int \sqrt{a\left[\{(\sec[z])^2-1\}+1\right]}\,[\sec(z)]^2\,dz\right)}{\sqrt{b}}$$

Expand the right-hand side of the equation to reap the fruits of the substitution:

$$\Delta\ldots\ldots = \frac{a\left(\int [\sec(z)]^3\,dz\right)}{\sqrt{b}}$$

That looks ugly. If it was squared instead of cubed, it may crack immediately. This, however, is difficult. Maybe a hyperbolic substitution will work. Delete every equation with z in it. Try this: enter the equation $bx^2 = a(\sinh[z])^2$, isolate x, and Expand the result to get:

$$\Delta x = \frac{\sqrt{a}\sinh(z)}{\sqrt{b}}$$

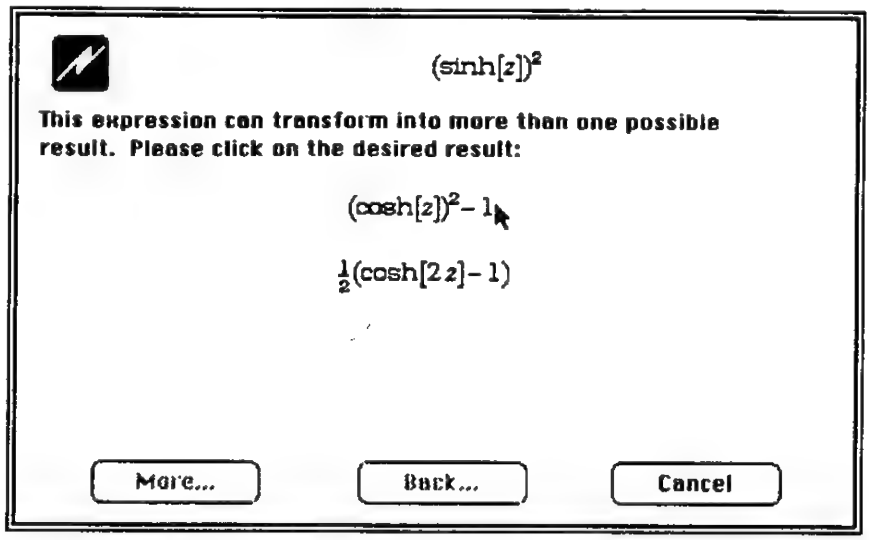
Substitute it into our integral,

$$\Delta\ldots\ldots = \int \sqrt{b\,x^2+a}\,dx$$

...and Expand the result to get:

$$\Delta\ldots\ldots = \frac{\sqrt{a}\left(\int \sqrt{a\,[\sinh(z)]^2+a}\,\cosh[z]\,dz\right)}{\sqrt{b}}$$

Collect inside the square root. Transform the $(\sinh[z])^2$ as before, which gives this dialog:



Click on the first result, which gives you:

$$\triangle\ldots\ldots = \frac{\sqrt{a}\left(\int \sqrt{a\left[(\cosh(z))^2-1\right]+a}\,\cosh[z]\,dz\right)}{\sqrt{b}}$$

Expand the right-hand side to get:

$$\triangle\ldots\ldots = \frac{\sqrt{a}\left(\int \cosh[z]\sqrt{a\,[\cosh(z)]^2}\,dz\right)}{\sqrt{b}}$$

...a much more promising integral. Now, all we need is an identity, one with sin (2*x*) or whatever, except for hyperbolics. They are similar to the identities for trigonometric functions, but the trigonometric identities are hard to remember anyway.

Select the whole square root:

$$\triangle\ldots\ldots = \frac{\sqrt{a}\left(\int \cosh[z]\sqrt{a\,[\cosh(z)]^2}\,dz\right)}{\sqrt{b}}$$

...and Expand to get:

$$\triangle \cdots\cdots = \frac{\sqrt{a}\left(\int \cosh[z]\left[\sqrt{a}\cosh(z)\right]dz\right)}{\sqrt{b}}$$

Simplify the entire right-hand side of the equation:

$$\triangle \cdots\cdots = \frac{a\left(\int [\cosh(z)]^2 dz\right)}{\sqrt{b}}$$

Select $(\cosh[z])^2$ and Transform it. (Use the second result when Theorist asks you which one you want.)

$$\triangle \cdots\cdots = \frac{a\left(\int \left[\frac{1}{2}\{\cosh(2z)+1\}\right]dz\right)}{\sqrt{b}}$$

Expand the entire right-hand side of the equation to get the answer:

$$\triangle \cdots\cdots = \frac{1}{2}\frac{az}{\sqrt{b}} + \frac{1}{4}\frac{a\sinh(2z)}{\sqrt{b}}$$

The integral is solved but the job is not done until the paperwork is done. We still have two levels of substitution to crawl back through. We start by substituting back to *x*.

Locate the following equation (actually, any relation between *x* and *z* will suffice):

$$\triangle x = \frac{\sqrt{a}\sinh(z)}{\sqrt{b}}$$

Isolate *z*:

$$\triangle z = \operatorname{arcsinh}\left(\frac{\sqrt{b}\,x}{\sqrt{a}}\right)$$

Now, substitute it into the equation,

$$\triangle \cdots\cdots = \frac{1}{2}\frac{a\,\blacksquare}{\sqrt{b}} + \frac{1}{4}\frac{a\sinh(2\,\blacksquare)}{\sqrt{b}}$$

...which gives you:

$$\triangle \cdots\cdots = \frac{1}{2}\frac{a\operatorname{arcsinh}\left(\frac{\sqrt{b}\,x}{\sqrt{a}}\right)}{\sqrt{b}} + \frac{1}{4}\frac{a\sinh\left(2\operatorname{arcsinh}\left[\frac{\sqrt{b}\,x}{\sqrt{a}}\right]\right)}{\sqrt{b}}$$

Before we go any further, let's pound on that sinh of an arcsinh. Select the sinh[2...] as shown,

$$\triangle \cdots\cdots = \frac{1}{2}\frac{a\operatorname{arcsinh}\left(\frac{\sqrt{b}\,x}{\sqrt{a}}\right)}{\sqrt{b}} + \frac{1}{4}\frac{a\,\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare}{\sqrt{b}}$$

...then Transform it. (In the dialog, click on the result which *does not* contain exponentials.)

$$\triangle \cdots\cdots = \frac{1}{2}\frac{a\operatorname{arcsinh}\left(\frac{\sqrt{b}\,x}{\sqrt{a}}\right)}{\sqrt{b}} + \frac{1}{4}\frac{a\left(2\frac{\sqrt{b}\,x\cosh\left[\operatorname{arcsinh}\left(\frac{\sqrt{b}\,x}{\sqrt{a}}\right)\right]}{\sqrt{a}}\right)}{\sqrt{b}}$$

Wow. This turned into a sinh times a cosh, and Auto Simplify annihilated the sinh of the arcsinh and made it *x*. Using techniques learned previously, we can deal with the cosh of the arcsinh. Enter a new expression:

y is a wildcard variable.

$$\square(\sinh[y])^2$$

Then, Transform it into:

$$\Delta(\sinh[y])^2 = (\cosh[y])^2 - 1$$

Isolate cosh y to get:

$$\Delta\cosh(y) = \sqrt{(\sinh[y])^2 + 1}$$

Substitute the above equation into the equation,

$$\Delta\cdots = \frac{1}{2}\frac{\alpha \operatorname{arcsinh}\left(\frac{\sqrt{b}\,x}{\sqrt{\alpha}}\right)}{\sqrt{b}} + \frac{1}{4}\frac{\alpha\left(2\frac{\sqrt{b}\,x\cosh\left[\operatorname{arcsinh}\left(\frac{\sqrt{b}\,x}{\sqrt{\alpha}}\right)\right]}{\sqrt{\alpha}}\right)}{\sqrt{b}}$$

...and the result is automatically simplified to:

$$\Delta\cdots = \frac{1}{2}\frac{\alpha \operatorname{arcsinh}\left(\frac{\sqrt{b}\,x}{\sqrt{\alpha}}\right)}{\sqrt{b}} + \frac{1}{2}\sqrt{\alpha}\sqrt{\left(\frac{\sqrt{b}\,x}{\sqrt{\alpha}}\right)^2 + 1}\,x$$

We can simplify this even further. Collect everything inside of the big square root, then Expand the right-hand side of the resulting equation:

$$\Delta\cdots = \frac{1}{2}\frac{\alpha \operatorname{arcsinh}\left(\frac{\sqrt{b}\,x}{\sqrt{\alpha}}\right)}{\sqrt{b}} + \frac{1}{2}\sqrt{b x^2 + \alpha}\,x$$

Nice and simple. Now we are ready to substitute back to θ. Get the equation $x = \sin\theta$ and Substitute into the equation,

$$\Delta\cdots = \frac{1}{2}\frac{\alpha \operatorname{arcsinh}\left(\frac{\sqrt{b}\,\square}{\sqrt{\alpha}}\right)}{\sqrt{b}} + \frac{1}{2}\sqrt{b\square^2 + \alpha}\,\square$$

...which yields:

$$\Delta \cdots = \frac{1}{2}\frac{\alpha \operatorname{arcsinh}\left(\frac{\sqrt{b}\sin[\theta]}{\sqrt{\alpha}}\right)}{\sqrt{b}} + \frac{1}{2}\sqrt{b(\sin[\theta])^2+\alpha}\sin(\theta)$$

...which is the final answer.

Verifying the Answer

Whenever you do an integral like this, you should check the answer to make sure it is correct. Copy the right hand side of the equation and create a new assumption. Type "d(" then do a Paste:

$$\square\, d\left(\frac{1}{2}\frac{\alpha \operatorname{arcsinh}\left[\frac{\sqrt{b}\sin(\theta)}{\sqrt{\alpha}}\right]}{\sqrt{b}} + \frac{1}{2}\sqrt{b[\sin(\theta)]^2+\alpha}\sin[\theta]\right)$$

Simplify it to process the derivative. The right hand side looks like this:

$$\frac{1}{2}\frac{\sqrt{\alpha}\cos(\theta)}{\sqrt{\left(\frac{\sqrt{b}\sin[\theta]}{\sqrt{\alpha}}\right)^2+1}}d\theta + \frac{1}{2}\frac{b\cos(\theta)(\sin[\theta])^2}{\sqrt{b(\sin[\theta])^2+\alpha}}d\theta + \frac{1}{2}\sqrt{b(\sin[\theta])^2+\alpha}\cos(\theta)d\theta$$

Collect inside the biggest square root, on the denominator of the first term on the right:

$$\Delta d\cdots = \frac{1}{2}\frac{\sqrt{\alpha}\cos(\theta)}{\sqrt{\frac{\alpha\left(\frac{\sqrt{b}\sin[\theta]}{\sqrt{\alpha}}\right)^2+\alpha}{\alpha}}}d\theta + \cdots$$

Expand the first term on the right-hand side of the equation, and many things cancel out.

$$\Delta d\cdots = \frac{1}{2}\frac{\alpha\cos(\theta)}{\sqrt{b(\sin[\theta])^2+\alpha}}d\theta + \cdots$$

Collect the whole right side of the equation:

$$\Delta d\cdots = \frac{1}{2}\frac{\left(2b\left[\sin(\theta)\right]^2+2a\right)\cos(\theta)}{\sqrt{b\left(\sin[\theta]\right)^2+a}}\,d\theta$$

It looks as though the top and bottom will cancel out some. In the numerator, select $2b(\sin[\theta])^2+2a$ and do a Collect:

$$\Delta d\cdots = \frac{1}{2}\frac{\left(2\left[b\left(\sin(\theta)\right)^2+a\right]\right)\cos(\theta)}{\sqrt{b\left(\sin[\theta]\right)^2+a}}\,d\theta$$

Simplify the right side of the equation. You get:

$$\Delta d\cdots = \sqrt{b\left(\sin[\theta]\right)^2+a}\,\cos(\theta)\,d\theta$$

...which was our original integrand. Our answer was right.

Creating a Transformation Rule

If you find yourself having to do the same integrals over and over many times, you could make a rule. Locate the equation where the integral was finally solved:

$$\Delta\int\sqrt{b\left(\sin[\theta]\right)^2+a}\,\cos(\theta)\,d\theta = \frac{1}{2}\frac{a\,\operatorname{arcsinh}\left(\frac{\sqrt{b}\,\sin[\theta]}{\sqrt{a}}\right)}{\sqrt{b}}+\frac{1}{2}\sqrt{b\left(\sin[\theta]\right)^2+a}\,\sin(\theta)$$

Click anywhere within the equation. Then, from the **Input** menu, select **Transform Rule**, which makes this rule:

The rule is shown broken into two lines so it fits on this page; in your notebook, it appears on one line.

Upon [Transform] transform $\int\sqrt{b\left(\sin[\theta]\right)^2+a}\,\cos(\theta)\,d\theta$

into $\frac{1}{2}\frac{a\,\operatorname{arcsinh}\left(\frac{\sqrt{b}\,\sin[\theta]}{\sqrt{a}}\right)}{\sqrt{b}}+\frac{1}{2}\sqrt{b\left(\sin[\theta]\right)^2+a}\,\sin(\theta)$.

There are some changes you may want to make to the rule before implementing it. For instance, you may want it to be done upon Simplify instead of upon Transform. Also, you may want to change the θ to a wildcard variable such as **t**, so it works for similar equations, instead of just situations where your variable happens to be θ. While you are at it, change the a and b to **a** and **b**. Note that **a** and **b** match only constants, whereas the **t** matches variables.

BESSEL'S DIFFERENTIAL EQUATION

In this tutorial, we attempt to solve Bessel's differential equation by using power series (i.e., finding its polynomial expansion).

Start by opening the notebook "Finite & Infinite Series" in the Mathematics folder. We'll use one of the rules in that notebook in the "Joining, Unraveling and Shifting" bundle.

Enter Bessel's differential equation:

$$\left(x^2\left[\frac{\partial}{\partial x}\right]^2 + x\frac{\partial}{\partial x} + x^2 - \nu^2\right) y = 0$$

Option-N without Shift does something confusing with the Macintosh keyboard—avoid it.

You can enter ν (Greek nu) by pressing Option-Shift-N. In this equation, y is a function of x, and ν is a constant, usually an integer. Clarify, making ν a constant.

Start out by assuming a solution in the form of a generic power series for y in terms of x:

$$\square y = \sum_{k=0}^{\infty} b_k x^k$$

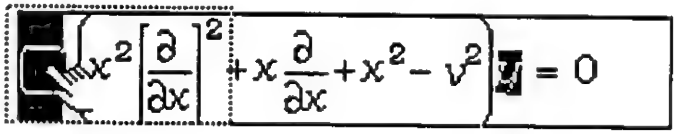
Clarify the notebook and declare b as a constant, and make ∞ predefined. k should already be defined correctly. It cannot be a constant, otherwise it would not be allowed in the summation.

We have to convince Theorist that k will not change in value as x changes in value (as if perhaps k = floor[x]). In addition, we need to specify that b_k will not change in value either; if k can change, then b_k can, too. In order to make this work, we need to specify that k and b_k are independent of x. The easiest way is to declare that they are independent of every other variable.

From Theorist's **Input** menu, choose **Independence Decl.** With the question mark selected, type "`k,b_k`". Change this declaration's pop-up menu to "all variables".

This way, the x's and the derivatives can pass through the Σ.

Take the summation equation and substitute it into the differential equation by clicking and dragging:



...to get the following result:

$$\triangle\left(x^2\left[\frac{\partial}{\partial x}\right]^2+x\frac{\partial}{\partial x}+x^2-v^2\right)\sum_{k=0}^{\infty}b_k x^k=0$$

Expand the result three times.

$$\triangle x^2\sum_{k=0}^{\infty}b_k k^2 x^{k-2}+\cdots\cdots=0$$

This answer is large. We want some of the terms to cancel out, but we have to help Theorist. Situations where there are things multiplied together that don't commute (such as the summations and variables here) can be confusing to Expand and Simplify, that already have their hands full doing other things. To prevent accidents, they conservatively prohibit you from doing things that may not be correct. In such situations, you have to help it along by using Commute. Commute is simpler, because it only has to worry about a few factors at a time, but it more fully understands what commutes with what, and it allows you the full freedom to do what you need to do.

As a shortcut to dragging to select x^2, click just below the "2".

Now, take x^2 from the first term and move it (by Command-dragging) to the right, past the Σ sign, so that it is next to the x^{k-2}. You may have to drag carefully to get the location you want to highlight.

Option-Command-dragging does the manipulation in place.

Do the same for each term that starts with an x or an x^2. Hold down both Command and Option as you move it to the right, past the Σ next to the x^k, x^{k-1}, or x^{k-2}. After the four moves, you end up with:

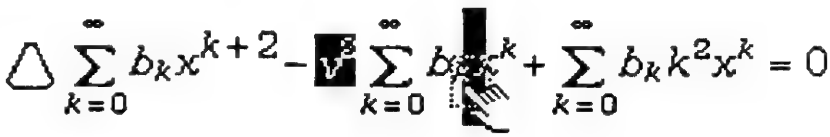
$$\Delta \sum_{k=0}^{\infty} b_k k^2 x^2 x^{k-2} - \sum_{k=0}^{\infty} b_k k x^2 x^{k-2} + \sum_{k=0}^{\infty} b_k k x x^{k-1} + \sum_{k=0}^{\infty} b_k x^2 x^k - \nu^2 \sum_{k=0}^{\infty} b_k x^k = 0$$

Simplify the whole equation. The x's unify, and two large terms cancel each other, to yield:

$$\Delta \sum_{k=0}^{\infty} b_k x^{k+2} - \nu^2 \sum_{k=0}^{\infty} b_k x^k + \sum_{k=0}^{\infty} b_k k^2 x^k = 0$$

Since this is true for all x, our aim is to equate coefficients. We want to collect the second two terms together. If you try it, you'll notice that nothing happens. Once again, Collect must be helped along by Commute. When non-commutativity raises its ugly head, Collect can only collect factors that are on the far left or far right.

To help it along, select the ν^2 in the middle term and Command-drag it to the right of the b_k, in the same place that the k^2 is in the last term, as shown:



...which gives you:

$$\Delta \sum_{k=0}^{\infty} b_k x^{k+2} - \sum_{k=0}^{\infty} b_k \nu^2 x^k + \sum_{k=0}^{\infty} b_k k^2 x^k = 0$$

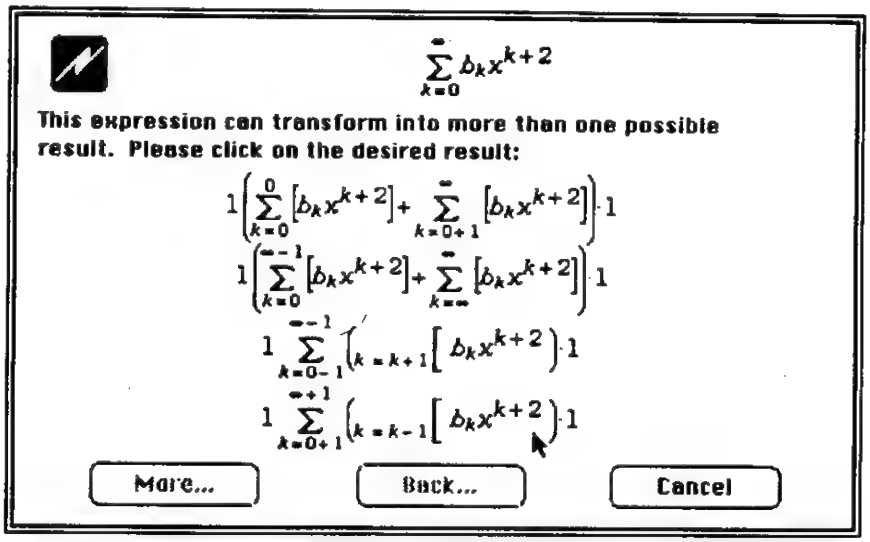
Next, Collect the last two terms:

$$\Delta \sum_{k=0}^{\infty} b_k x^{k+2} + \sum_{k=0}^{\infty} (k^2 - \nu^2) b_k x^k = 0$$

If you didn't open this notebook, the next step is not going to work.

Then, change the x^{k+2} in the first term to x^k so that the x's line up. To shift the whole summation over by two, we'll make use of a transformation rule in the "Finite & Infinite Series" notebook.

Select the whole first term and choose **Transform** from the **Manipulate** menu. The Transform dialog displays four possible results to choose from. You want the one that has the "$k = k - 1$" in it, the bottom rule:



Once you get an answer, keep the selection (which is the first term) and Transform again to move it over one more step. You should have:

$$\triangle \sum_{k=2}^{\infty} b_{k-2} x^k + \sum_{k=0}^{\infty} \left(k^2 - v^2\right) b_k x^k = 0$$

Instead of typing these from scratch, you can copy and paste pieces from the last conclusion.

Next, generate the recurrence relations, one for each value of k, by creating the following new assumptions:

$$\square \left(k^2 - v^2\right) b_k = 0$$
$$\square\, b_{k-2} + \left(k^2 - v^2\right) b_k = 0$$

The first is only used for k values of 0 and 1. Enter new assumptions for $k = 0$ and $k = 1$. Substitute each of them, one at a time, into the $(k^2 - n^2)\, b_k = 0$ equation to generate the equations for 0 and 1.

$$\triangle -b_0 v^2 = 0$$
$$\triangle \left(-v^2 + 1\right) b_1 = 0$$

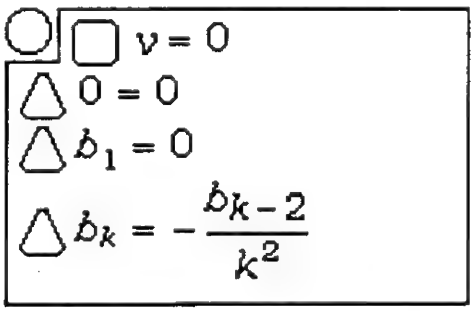
Make sure Auto Casing is off and Isolate b_k from the other new equation.

$$\triangle b_k = -\frac{b_{k-2}}{k^2 - v^2}$$

The three conclusions we have here are all we need to generate all of the coefficients for our polynomial.

The letter *v* won't work; you must use the Greek letter nu, *v*. Press Option-Shift-N.

Let's first examine the case where $v = 0$. Make a new case theory, and enter $v = 0$ in its blank assumption. Substitute this into each of our three magic conclusions from the last step.



You will notice that we get a firm message that b_1 is zero, but nothing about b_0. It is arbitrary. Enter $b_0 = 1$ in a new assumption within this case theory. Also enter $k = 2$ in a new assumption within this case theory.

Substitute $k = 2$ into our equation for b_k. Change $k = 2$ to $k = 3$ and repeat. Keep on generating equations for the b_k's until you get to 6. (You can skip the odd ones because they all end up as zero.)

$$\triangle b_k = -\frac{b_{k-2}}{k^2}$$

$$\square\ b_2 = -\tfrac{1}{4} b_0$$

$$\square\ b_4 = -\tfrac{1}{16} b_2$$

$$\triangle b_6 = -\tfrac{1}{36} b_4$$

Now, substitute equations around until you have numerical values for all of them.

$$\square\ b_2 = -\frac{1}{4}b_0$$
$$\triangle\ b_2 = -\frac{1}{4}$$
$$\square\ b_4 = -\frac{1}{16}b_2$$
$$\triangle\ b_4 = \frac{1}{64}$$
$$\triangle\ b_6 = -\frac{1}{36}b_4$$
$$\triangle\ b_6 = -\frac{1}{2304}$$

A check with *Handbook of Mathematical Functions* (Abramowitz and Stegun, Dover Publications) tells us that we have found the coefficients for $J_0(x)$, the most well-known of the Bessel functions. If we were to substitute in $\nu = 1$, we could get the coefficients for $J_1(x)$, which only has coefficients for odd values of k. Other values for ν would yield the full set.

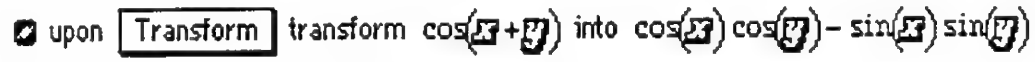
It turns out that there's another solution to this equation that cannot be modeled as a power series because it has a singularity at $x = 0$. These functions are the Bessel functions of the second kind, known collectively as $Y_n(x)$. Together the J's and the Y's and their linear combinations constitute a full space of solutions.

Modeling these functions is a tricky business. In the "Mathematics" folder's "Special Functions" folder, the file "Bessel J, Y, I, K" has well-engineered expressions for modeling the Bessel functions for all real values of x.

Trans-formation Rules

TRANSFORMATION RULES

Because there is an infinite number of mathematical formulas, Theorist provides a set of programmable rules. You have full power to program these rules and apply them at your convenience. Here is an example of a typical rule:



The rule is self explanatory: upon the **Transform** command from the **Manipulate** menu it transforms the cosine of the sum of x and y according to a well-known identity.

There are two things to notice in the above example: the use of the wildcard variables and the Transform pop-up menu. Wildcard variables allow you to use expressions as arguments. For example, if you have:

$$\square\ \cos(3t+2\pi)$$

...Theorist treats $3t$ as x and 2π as y, transforming them into:

$$\triangle\ \cos(3t+2\pi)=\cos(3t)\cos(2\pi)-\sin(3t)\sin(2\pi)$$

Be careful how you use wildcard variables; they have to correspond to the class of entities they represent:

Wildcard Variable Guidelines

- a through h match any constants, excluding matrices
- t through n match any positive integer: 3, 1042, but no names
- o through q match any expression including matrices and operators
- r through z match any expression of class variable or constant

The Transform pop-up menu has two other options, Simplify and Expand. With this menu, you can choose whether the rule is activated when you do a Transform manipulation, when you Expand, or when you Simplify.

Making a Rule Method #1

At this point you know enough preliminaries, so you can enter some rules. First, choose **Transform Rule** from the **Input** menu. You get:

upon [Transform] transform ? into ?

You can insert the wildcard variables from the *x* palette's special pop-up palette for wildcards.

Enter the first part of the rule,

$$\int (\sin [a\,x])^2 dx$$

...into the left ?, then select the right ? and enter the second part of the rule,

$$\frac{x}{2} - \frac{\sin(2\,a\,x)}{4\,a}$$

In this case, a is a constant and x is a variable.

Making a Rule Method #2

There is another way of teaching Theorist this rule. Delete the previous rule (to avoid confusion later) and enter the following equation.

$$\int (\sin[a\,x])^2 dx = \frac{x}{2} - \frac{\sin(2\,a\,x)}{4\,a}$$

Use Transform Rule from the Input menu, not Transform on the Manipulate menu.

Select the equation and issue the Transform rule command again. You get:

Upon [Transform] transform $\int (\sin[a\,x])^2 dx$ into $\frac{x}{2} - \frac{\sin(2\,a\,x)}{4\,a}$.

To test the rule, enter the expression:

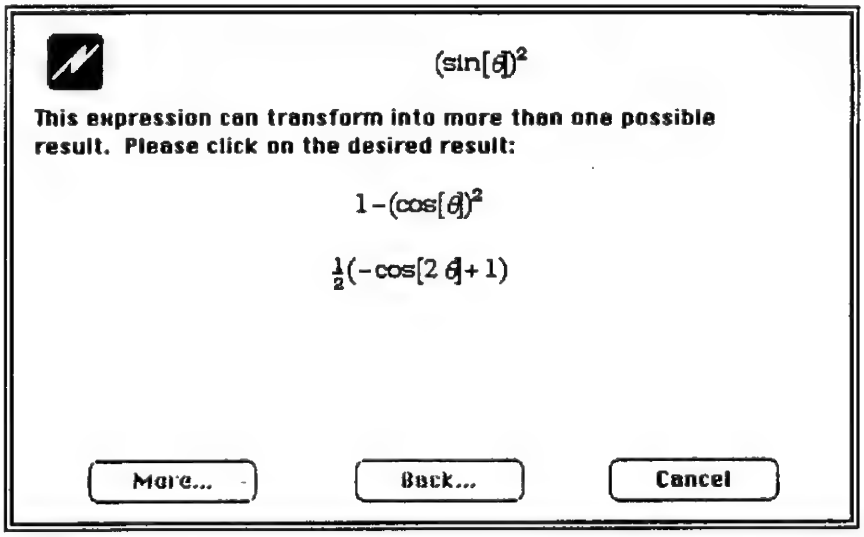
$$\int (\sin[5\,z])^2 dz$$

Click on the integral sign to select the expression and choose **Transform** from the **Manipulate** menu.

$$\int (\sin[5\,z])^2 dz = -\frac{1}{20}\sin(10\,z) + \frac{1}{2}z$$

If you prefer, use the Transform pop-up menu to change the rule setting so the transformation is performed whenever you Simplify. Do this now. Click on the integral sign and choose **Simplify** from the **Manipulate** menu, and Theorist gives you the correct result, same as the last one.

Whenever you give Theorist more than one transform for the same thing, it puts up the following dialog:



...from which you can choose the appropriate answer. (The previous tutorial, the Complicated Integral, made use of this feature.)

CHEBYSHEV POLYNOMIAL GENERATION

In this tutorial, we will learn about wildcard variables, transformation rules, and some of the power behind Theorist's manipulation commands.

Theory Digression
(Do not start yet)

Chebyshev (sometimes spelled Tchebychef) polynomials are a set of orthogonal polynomials used to solve differential equations. The norm through which they are orthogonal is:

$$\int_{-1}^{1} \frac{T_n(x)\,T_m(x)}{\sqrt{1-x^2}}\,dx$$

Given that:

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$$

...we will derive them all.

Now you can start.

Start with a new notebook and save it as "Chebyshev". Save it often as you do this tutorial.

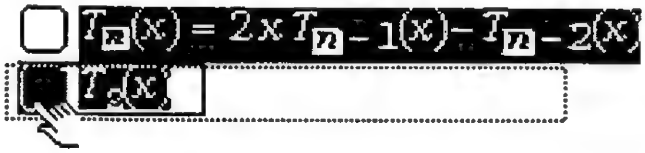
Use Command-L or "_" (underscore) to enter subscripts.

First, enter the three equations shown above. When you enter the *n*'s, don't just type "**n**"—instead, type "**?n**" or choose $\boldsymbol{n}$ from the variables palette. The $\boldsymbol{n}$ is a wildcard variable which matches any integer. We need this because the third equation, the recurrence relation, is used later to generate others. After you enter the equations, you have:

$$\square\ T_0(x) = 1$$
$$\square\ T_1(x) = x$$
$$\square\ T_{\square}(x) = 2x\,T_{\square-1}(x) - T_{\square-2}(x)$$

Clarify the notebook and Declare *T* as a **Function**.

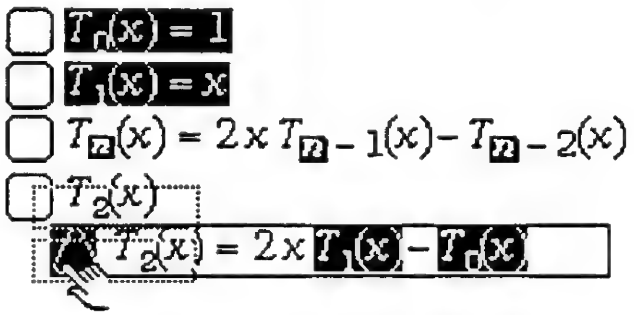
The next step is to derive the answer for $T_2(x)$. First, enter $T_2(x)$, then substitute the recurrence relation over it:



Note that the answer matches:

$$\triangle\ T_2(x) = 2x\,T_1(x) - T_0(x)$$

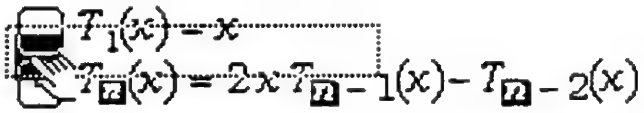
The **n** matches the 2. Since this gives T_2 in terms of T_1 and T_0, we need to substitute them in, too. Select the first two equations by Shift-clicking, then Substitute them over your new equation.



This yields the answer:

$$\triangle\ T_2(x) = 2x^2 - 1$$

Now, move the last conclusion up under the T_1 equation.



Delete the intermediate equation, which is now the last one, at the bottom of the notebook. Change the last expression from $T_2(x)$ to $T_3(x)$ for the next polynomial.

Now, repeat the process. Expand the results. By mathematical induction, you can generate all Chebyshev polynomials:

You may have to use Expand to get some of them to look like this.

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_2(x) = 2x^2 - 1$$
$$T_3(x) = 4x^3 - 3x$$
$$T_4(x) = 8x^4 - 8x^2 + 1$$
$$T_n(x) = 2x\,T_{n-1}(x) - T_{n-2}(x)$$

Yes, the mathematician says you can generate all Chebyshev polynomials that way. The engineer, however, says "I'm sick of doing this. Isn't there an easier way?" The answer is "Yes".

Delete your hard-earned equations, except for the original three:

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_n(x) = 2x\,T_{n-1}(x) - T_{n-2}(x)$$

Select each of the three by Shift-clicking the equal signs:

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_n(x) = 2x\,T_{n-1}(x) - T_{n-2}(x)$$

Choose **Transform rule** from the **Input** menu. Theorist generates three transformation rules. Use the pop-up menus in the rules and change them from **Transform** to **Simplify**:

upon [Simplify] transform $T_0(x)$ into 1
$T_1(x) = x$
upon [Simplify] transform $T_1(x)$ into x
$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$
upon [Simplify] transform $T_n(x)$ into $2xT_{n-1}(x) - T_{n-2}(x)$

Now, click some white space outside the propositions so nothing is selected. Press Return, and enter $T_7(x)$. Select and Simplify it. With the Auto Simplify preference on, Theorist repeatedly simplifies the expression, iterating the transformations up to ten times before quitting. (If you have to calculate $T_{25}(x)$, it's not much more work to simplify three times.) Expand the result, and you get the final answer:

$$\triangle T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$$

Note that Theorist looks for transformation rules top-down, so it is important to have the rules for specific cases before those for general cases. When general cases match first, specific cases are not applied, resulting in things like $T_{-1}(x)$.

Exercise

The Legendre polynomials can be generated from the rules:

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_n(x) = \frac{2n-1}{n} xP_{n-1}(x) - \frac{n-1}{n} P_{n-2}(x)$$

Go for it.

SUPPORT

Technical Support

There may come a time during your experience with Theorist when you have a question about some technical aspect of its operation. If this happens, your first instinct may be to call or write for help. Before you do this, however, make sure you check out your first source of technical support—the manuals. The Theorist Learning Guide and Reference Manual are designed to provide you with a complete reference for the software. Answers to most questions about the program can be found in the Reference Manual. If, after searching through the manuals, you still cannot find an answer to your question, you may write to:

Prescience Corporation
Attention: Technical Support
939 Howard Street
San Francisco, CA 94103

or you may telephone (415) 543-2252 and ask for Theorist Technical Support. FAX us at (415) 882-0530.

Please include your serial number with all correspondence and be prepared to give it to the technical support specialist when you call.

Customer Service

You can also use the above address and telephone number to obtain support for non-technical matters, or customer service. Direct all of your non-technical questions and business-related matters to the above address or phone number, but to the attention of Customer Service.

Owner Registration Card

Please complete your owner registration card, included in your Theorist package, and send it to Prescience as soon as possible. Being a registered owner entitles you to free technical support. It also means you will receive *The Math Axis* newsletter, with tips on using Theorist and Expressionist, and the latest news about upgrades and new products. Registered users also get big discounts on new products. So please send it in!

BUG REPORT FORM

Photocopy this form and use it to report bugs that you find in Theorist, or inaccuracies in the documentation. We thank you in advance for your help in making Theorist a better product.

You must be a registered user to receive technical support.

Serial number (10-digit): _____

_____
Name

_____ _____
Address Phone or Fax number(s)

_____ _____
City, State, and Zip or Postal Code Country

The problem is with: (check all that apply, describe on separate paper)
- ❑ Using Theorist by itself
- ❑ Using Theorist with other programs
- ❑ The documentation; manual title and page: _____
- ❑ Printed output; printer make and model: _____

I discovered the problem while using: (check all that apply)

❑ Plus	❑ Classic	❑ SE	❑ SE/30	❑ Portable
❑ II	❑ IIx	❑ IIcx	❑ IIci	❑ IIfx
❑ IIsi	❑ LC			

❑ Other Macintosh model:

(Theorist does not run on the Mac 128K, 512K, 512KE, XL, or the Lisa.)

Please describe the bug in as much detail as possible (on a separate sheet of paper, or the back of this page). What goes wrong? What were you trying to do when the bug appeared? Are you using MultiFinder? How much RAM does your machine have? (Theorist requires at least one megabyte.) What is your hardware setup (RAM disks, accelerator boards)? Did the bug appear just once, or repeatedly?

We are also open to general (and specific) suggestions. Write us a letter; tell us what you'd like to see in the program. (No guarantees.)

Send to: Prescience Corp., 939 Howard St., San Francisco, CA 94103
FAX to: (415) 882-0530

ACKNOWLEDGEMENTS

When I wrote Expressionist, I was all by myself. So I got a lot of sympathy and help from my friends. But becuase Expressionist has been such a success, I've been able to build a small company that supports my user interface dreams. The people who have joined me at Prescience have time and again worked way beyond the call of duty. I would especially like to thank Mike Robertson, Erik Warren, Laura Wallrath, and Chester Hurtado, who have worked especially hard to make this company a success.

As with Expressionist, Theorist is largely a product of the feedback I received from beta testers. It's been long road, and only a few of you have stayed with me for the years that it took to write this program, but you all contributed to the process. My sincere thanks to each of you.

I'd also like to thank Cara for putting up with me as my insanity escalated to arbitrarily high levels as the release date became arbitrarily close.

In addition, I would like to thank Mike Goddu, who made it possible for this business misfit to build a million-dollar company.

Finally, I would like to thank all of the people who built the many foundations that my work rests upon, including the symbolic algebra pioneers of the 1960's and 1970's as well as the wonderful folks who developed the tools for a truly modern user interface at Xerox PARC and Apple.

—Allan Bonadio
San Francisco
September 1990

ABOUT THE COMPANY

Prescience (pronounced PRĒ-*shenœ*) Corporation began as Allan Bonadio Associates, which was founded in July 1984 to develop graphical user interface software for the microcomputer and workstation markets.

ABA developed MacSpell+ in conjunction with Level Four Interactive, Inc. and Creighton Development, Inc. This spelling-checking program reached number six on the Soft-Sel national best seller list and remained on the list for over six months.

The company released the Macintosh equation editor Expressionist in March 1987. Used to generate equations for technical desktop publishing, the program extends the Macintosh user interface to equations and allows WYSIWYG editing of expressions in a point-and-click manner. Expressionist has consistently been on MacAmerica's best-seller list and has become the standard equation editor for the Macintosh.

ABA incorporated as Prescience Corporation in May 1989. From its office in downtown San Francisco, the company publishes Expressionist and contemplates new products.

President Allan Bonadio earned a B.S. in Applied and Engineering Physics from Cornell University, where he minored in computer science and psychology. A programmer for eighteen years, he has worked as a programmer, technical writer, and user interface consultant to Apple Computer, Hewlett-Packard, NeXT, Inc., Digital Equipment Corporation, and other companies. He enjoys socializing, vacationing, cooking, psychology, scuba diving and the history of Indo-European languages. As the head programmer, he is currently developing new products and augmenting existing products for Prescience Corporation.

COLOPHON

Design

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Allan Bonadio
Michelle Miller

Notebooks

Allan Bonadio
Alex Plotitsa

Documentation

Allan Bonadio
BC Crandall
Erik Warren

Documentation Tools

Apple Macintosh, LaserWriter IINTX, Theorist, Expressionist, Canvas, DeskPaint, Exposure, FlashIt, and FullWrite Professional.

Packaging design

Clennon Associates (Pacifica, California)

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